CONTROLLER DESIGN USING COMBINATION OF SYMBOLIC AND NUMERIC CALCULATIONS IN MAPLE

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ABSTRACT
The paper shows how it is possible to combine symbolic computations with numerical methods in order to achieve the controller action under specified conditions. The corresponding calculations are based on the nonlinear state space model resulted from dynamics decomposition. The controlled system is represented by the triple integrator with input saturation. The design combines the well known time optimal control with the linear pole assignment control, i.e. the control consist of n phases similar to the time optimal control, however the transients between these phases are "smooth" and the dynamics of the transients is given by the closed loop poles. For modeling and simulations the computer algebra system Maple has been used.

Keywords: nonlinear model, symbolic calculations, pole assignment control, time optimal control

1. INTRODUCTION

Recent decade in the control design is characteristic by a revival of theory of constrained systems. The minimum time control, which was dominating the control design from late 40-ties to the beginning of 70-ties in the 20th century (see e.g. Pavlov 1966; Athans and Falb 1966) is, however, replaced by several new approaches as the predictive control (see e.g. Bemporad et al. 2002; El-Farra et al. 2003), different anti-windup solutions, positive invariance sets, etc. Motivation comes from different fields – from the traffic control, robot control, control of unstable systems, etc. A common feature of the new approaches is that they are rather complex - even in the case of simple control problems. So, they are not easy to understand and to apply. Traditionally, the engineering community preferred simpler solutions, as e.g. that one proposed by Kiendl and Schneider (1972), which was later used in robot control (Kunze 1984; Patzelt 1981).

Parallel to this, family of new not yet widely known solutions (Huba 1994; Huba and Bisták 1995; Huba and Bisták 1999; Huba 2006) was developed also thanks to new possibilities given by the modelling and simulation software tools. They are relatively simple for understanding, easy to implement and so appropriate also for extremely fast application and easy to tune by a procedure that can be considered as a generalization of the well-known method by Ziegler and Nichols. For the controller design purposes we usually use the computer algebra system Maple (mainly for symbolic calculations) beside the well-known Matlab (suitable for numeric methods and simulations). The whole presented design is carried out in the Maple environment.

In this paper, the family of the already known approaches is extended by the control design, which is based on the decomposition of the closed loop dynamics into particular modes defined by chosen real closed loop poles. This decomposition results in nonlinear state space model that is built in the Maple environment. Then the control law is calculated after localizing the current state in the nonlinear model and is related to the distance from the corresponding target object.

2. LINEAR POLE ASSIGNMENT CONTROL BASED ON MODES DECOMPOSITION

Let us consider the 3[rd] order integrator

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \quad \begin{matrix} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = u \end{matrix} \quad \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{1}$$

whereby $\mathbf{x}$ is the representative point, $\mathbf{b}$ is the input vector and $\mathbf{A}$ is the system matrix.

The linear pole assignment controller fulfills three rules:

1. It decreases the distance of the representative point from the plane.
2. It decreases the oriented distance of the representative point lying in that plane from the line.
3. Along that line the controller decreases the oriented distance from the origin.

Let us consider closed loop system with the following poles $\alpha_3 < \alpha_2 < \alpha_1 < 0$. Since the eigenvectors

$$\mathbf{v}_i = [\alpha_i \mathbf{I} - \mathbf{A}]^{-1}\mathbf{b} = \begin{bmatrix} \frac{1}{\alpha_i^{\ 3}} & \frac{1}{\alpha_i^{\ 2}} & \frac{1}{\alpha_i} \end{bmatrix}^T \tag{2}$$

are not collinear, they form a basis, which can be used for expressing any state as a sum of three modes

$$\mathbf{x} = q_1\mathbf{v}_1 + q_2\mathbf{v}_2 + q_3\mathbf{v}_3\,,\, q_1, q_2, q_3 \in R \tag{3}$$

Then one can write

$$\begin{aligned} \dot{\mathbf{x}} &= \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2 + \dot{\mathbf{x}}_3 = \\ &= \mathbf{A}\mathbf{x} + \mathbf{b}u = \\ &= \mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) + \mathbf{b}(u_1 + u_2 + u_3) \end{aligned} \tag{4}$$

After denoting $\mathbf{x}_i = q_i \mathbf{v}_i$ each subsystem can be expressed as

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{A}\mathbf{x}_1 + \mathbf{b}u_1 = \alpha_1 \mathbf{x}_1 \\ \dot{\mathbf{x}}_2 &= \mathbf{A}\mathbf{x}_2 + \mathbf{b}u_2 = \alpha_2 \mathbf{x}_2 \\ \dot{\mathbf{x}}_3 &= \mathbf{A}\mathbf{x}_3 + \mathbf{b}u_3 = \alpha_3 \mathbf{x}_3 \end{aligned} \tag{5}$$

So the 3rd order dynamics can be decomposed into three 1st order ones. The appropriate interpretation and the appropriate choice of the oriented distance measurement from the representative point $\mathbf{x}$ to the plane (or the line) lead to three control phases well known from the time optimal control. All of the coordinates $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ are equivalent in this linear case. Let us assign following:

1. The 1st equation describes the transient into the origin along the line given by $\mathbf{v}_1$. The dynamics is given by α_1.
2. The 2nd equation describes the transient in the plane given by $\mathbf{v}_1, \mathbf{v}_2$ to the line given by $\mathbf{v}_1$ measuring oriented distance in the direction of $\mathbf{v}_2$. The dynamics of the transient is given by the second pole α_2.
3. Similarly to 1. and 2. the 3rd equation describes the transient to the plane using the measurement direction of $\mathbf{v}_3$. The dynamics of this transient is given by α_3.

The total control signal is then the following sum

$$u = \sum_{i=1}^{3} u_i = \sum_{i=1}^{3} q_i \tag{6}$$

The control algorithm described above guarantees that all three control phases are running in parallel and this is equal to the result of the well-known Ackerman's formula. The Fig. 1 shows the new base described above. This new base creates the model for linear pole assignment control that is simply represented by three eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$. The control low can be computed by calculation of the distance between the representative point $\mathbf{x}$ and lines given by these three vectors. In the nonlinear case when the input constraints are present the model for pole assignment control will be more complicated because it consists of line segments (corresponding to linear control), curves a surfaces as it is shown below.

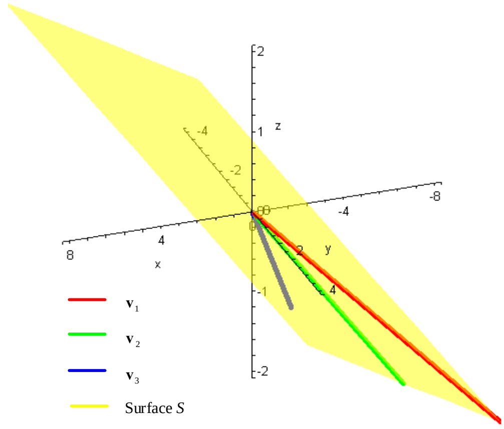

Figure 1: New basis of phase space

3. NON-LINEAR (CONSTRAINED) POLE ASSIGNEMNT CONTROL BASED ON MODES DECOMPOSITION

Let us consider a constrained control signal

$$u \in \langle U_1 \quad U_2 \rangle \tag{7}$$

The transient of the representative point with an initial state on the line given by the eigenvector $\mathbf{v}_1$ outside the proportional band does not follow this line. The particular subsystems do not change themselves only in their own coordinates $\mathbf{x}_i$, but also in the other subsystems coordinates. There is necessary to assign the coordinate to each of the control phases. The following assignment has been used:

1. The transient of the representative point in the phase space to the reference surface (*RS*) is given by $\alpha_3, \mathbf{v}_3$.
2. The transient along the *RS* to the reference curve (*RC*) is given by $\alpha_2, \mathbf{v}_2$.
3. The transient along the *RC* into the origin is given by $\alpha_1, \mathbf{v}_1$.

The control phases in this control algorithm combine the time optimal control with the linear pole assignment control. Each control phase is given by particular constraint of the control signal, but the transition between the control phases is described by the chosen corresponding closed loop pole. So each control phase is given by two parameters:

α_i - describes the dynamics of the transient of the representative point $\mathbf{x}_i$ inside the linear subsystem (proportional band)

t_i - the time, which is needed to reach the "linear" subspace under influence of limit control values..

One part of the final control phase (the linear one), corresponding to the transient into the origin, when the other coordinates $\mathbf{x}_3$ and $\mathbf{x}_2$ are zeroes, is given by the closed loop pole α_1 only along the line given by the eigenvector $\mathbf{v}_1$, where the coordinate

$$\mathbf{x}_1 \in \left\langle U_j \mathbf{v}_1 \quad U_{3-j} \mathbf{v}_1 \right\rangle \tag{8}$$

Let us assign the constraint of the control signal in the final control phase as U_j where

$$j = 1,2 \tag{9}$$

The linear control interval is restricted by

$$q_1 \in \left\langle U_j \quad U_{3-j} \right\rangle \tag{10}$$

The second part of the final control phase (non-linear) is described by the time t_1, that represents the time of the transient of the representative point using $u = U_j$ to the border points of the linear subsystem ${}^jX_1 = U_j \mathbf{v}_1$. The 2nd part is created by points $\mathbf{x}_1$ as the result of backward integration of (1) on the interval $t \in \langle 0 \quad t_1 \rangle$ using $u = U_j$ and starting from the point ${}^jX_1$. One gets

$$\mathbf{x}_1(t_1) = \begin{bmatrix} 1 & -t_1 & \frac{t_1^2}{2} \\ 0 & 1 & -t_1 \\ 0 & 0 & 1 \end{bmatrix} q_1 \mathbf{v}_1 + \begin{bmatrix} \frac{-t_1^3}{6} \\ \frac{t_1^2}{2} \\ -t_1 \end{bmatrix} U_j \tag{11}$$

So the $\mathbf{x}_1$ represents all the points of the 1st subsystem, and these points are given by the parameter q_1 for the points $t_1 = 0$, $\mathbf{x}_1 \in \left\langle U_j \quad U_{3-j} \right\rangle \mathbf{v}_1$ and also by $t_1 \neq 0$, $q_1 = U_j$ for the points outside the linear part (8). So let us introduce the following generalized denotation $\mathbf{x}_1(q_1, t_1)$, i.e. the points lying in the proportional band (the control signal is not saturated) are represented as $\mathbf{x}_1(q_1, 0)$ and the points lying outside (8) with saturated control signal are represented as $\mathbf{x}_1(U_j, t_1)$. The control signal for points in the proportional band $\mathbf{x}_1 = \mathbf{x}_1(q_1, 0) \in \langle U_1 \mathbf{v}_1 \quad 0 \rangle$ is $u_1 = z_1 \alpha_1$. The control signal for the points outside the proportional band $\mathbf{x}_1(U_j, t_1)$ is $u = U_j$. Such a representation of these points describes the transient along the *RC* with dynamics given by the closed loop pole α_1 with respecting the control constraints. The *RC* is invariant set of the system $\dot{\mathbf{x}}_1 = \mathbf{A}\mathbf{x}_1 + \mathbf{b}u_1$ with constrained control signal $u_1 \in \left\langle U_j \quad U_{3-j} \right\rangle$, i.e. after approaching the *RC*, the system remains on the *RC*.

In the second control phase is $\mathbf{x}_3 = 0$. There is the transient in the surface given by $\mathbf{x}_1$, $\mathbf{x}_2$ towards the *RC* (characterized by $\mathbf{x}_1$) described in this phase. The goal is to approach the *RC*, where $\mathbf{x}_2 = 0$. The dynamics of this transient is given by α_2 only in the proportional band of the second subsystem

$$\mathbf{x}_2 \in \left\langle 0 \quad (U_{3-j} - q_1) \right\rangle \mathbf{v}_2 \text{, i.e.}$$

$$q_2 \in \left\langle 0 \quad (U_{3-j} - U_j) \right\rangle \tag{12}$$

that gives the 1st part of the surface. The second part of the surface is given as the result of backward integration of (1) on the interval $t \in \langle 0 \quad t_2 \rangle$ starting from the points where the q_2 is saturated using $u = U_{3-j}$. Using generalized denotation of the *RC* one gets

$$\mathbf{x} = \mathbf{x}_2((U_{3-j} - q_1), t_2) + \mathbf{x}_1(q_1, t_1) =$$

$$= \begin{bmatrix} 1 & -t_2 & \frac{t_2^2}{2} \\ 0 & 1 & -t_2 \\ 0 & 0 & 1 \end{bmatrix} (\mathbf{x}_1(q_1, t_1) + q_2 \mathbf{v}_2) + \begin{bmatrix} \frac{-t_2^3}{6} \\ \frac{t_2^2}{2} \\ -t_2 \end{bmatrix} U_{3-j} \tag{13}$$

In the starting control phase the 3rd subsystem $\mathbf{x}_3$ is in the proportional band for

$$\mathbf{x}_3 \in \left\langle (U_j - q_1 - q_2)\mathbf{v}_3 \quad (U_{3-j} - q_1 - q_2)\mathbf{v}_3 \right\rangle \text{, i.e.}$$

$$q_3 \in \left\langle U_j - q_1 - q_2 \quad U_{3-j} - q_1 - q_2 \right\rangle \tag{14}$$

Similarly to the previous subsystems the result of backward integration on the interval $t \in \langle 0 \quad t_3 \rangle$ starting from the points where q_3 is saturated using $u = U_j$, gives the 3rd subsystem. Using generalized denotation $\mathbf{x}_2(q_2, t_2)$, the general point of the surface can be represented as $\mathbf{x}_1(q_1, t_1) + \mathbf{x}_2(q_2, t_2)$ and any point of the state space can be expressed using modes decomposition as

$$
\mathbf{x} = \mathbf{x}_3(q_3,t_3) + \mathbf{x}_2(q_2,t_2) + \mathbf{x}_1(q_1,t_1) =
$$

$$
= \begin{bmatrix} 1 & -t_3 & \frac{t_3{}^2}{2} \\ 0 & 1 & -t_3 \\ 0 & 0 & 1 \end{bmatrix} (\mathbf{x}_1(q_1,t_1) + \mathbf{x}_2(q_2,t_2) + \quad (15)
$$

$$
+ q_3\mathbf{v}_3) + \begin{bmatrix} \frac{-t_3{}^3}{6} \\ \frac{t_3{}^3}{2} \\ -t_3 \end{bmatrix} U_j
$$

Note that the whole state space can be described by $\mathbf{x}_1 = \mathbf{x}_1(q_1,t_1)$, $\mathbf{x}_2 = \mathbf{x}_2(q_2,t_2)$, $\mathbf{x}_3 = \mathbf{x}_3(q_3,t_3)$, using parameters $q_i, i = 1,2,3$ that describe the length of the vectors in the proportional bands of the subsystems and $t_i, i = 1,2,3$ describing these vectors outside theirs proportional bands (the control signal of the subsystem is saturated). The coordinates $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ create the model for pole assignment control of saturated system. The model is sophisticated because it consists of several segments of surfaces that are depicted below. The sequential choice of the coordinates of the subsystems guarantees, that the control signal is saturated only if the third subsystem is saturated also.

The control algorithm is the same as the linear one, but it depends on achieving the parameters $q_i, t_i, i = 1,2,3$ which are more difficult to obtain. The only way how to get them is to use symbolic calculations to derive the equation of a corresponding segment of surface and then to numerically evaluate the distance from the segment. The control is

$$
u = \sum_{i=1}^{3} u_i \tag{16}
$$

where u_i are control signals of particular subsystems. The *RS* is given by the points where $u_3 = 0$, but we divide it according to the parameters t_1, t_2, j. Let us denote following:

$RS^j{}_0$ - the part of *RS*, where $t_1 = 0, t_2 = 0$

$RS^j{}_1$ - the part of *RS*, where $t_1 > 0, t_2 = 0$

$RS^j{}_2$ - the part of *RS*, where $t_1 = 0, t_2 > 0$

$RS^j{}_{12}$ - the part of *RS*, where $t_1 > 0, t_2 > 0$

Following figures shows the *RC* and the *RS* for $j = 1$, $\alpha_1 = -0.5$, $\alpha_2 = -1$, $\alpha_3 = -2$, $u \in \langle -1 \quad 1 \rangle$.

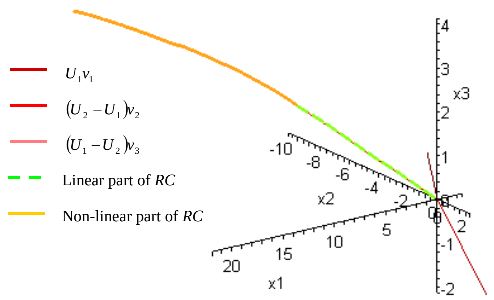

Figure 2: Eigenvectors $\mathbf{v}_i, i = 1,2,3$ and RC^1

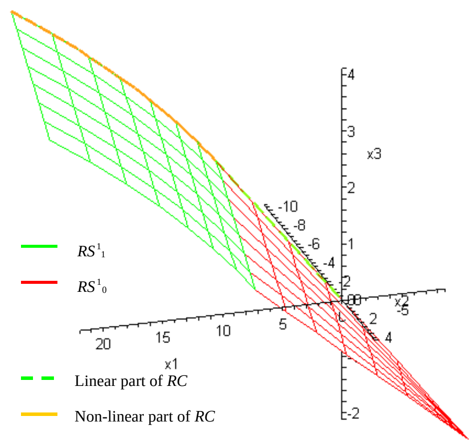

Figure 3: $RS^1{}_0$, $RS^1{}_1$

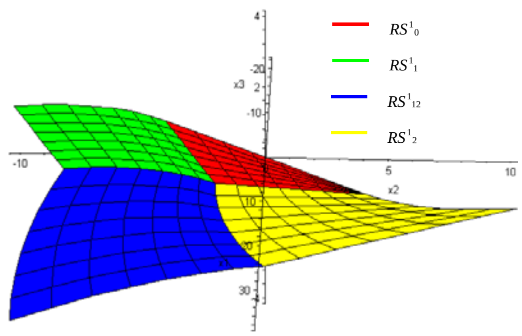

Figure 4: RS^1

The proportional band (*PB*) of the system (1) is given by points where $q_3 \in \langle U_1 - q_1 - q_2 \quad U_2 - q_1 - q_2 \rangle$. Let us denote $q_{3\min} = U_1 - q_1 - q_2, q_{3\max} = U_2 - q_1 - q_2$. Fig . 5 and Fig. 6 show the parts of *PB* corresponding to particular segments of *RS*.

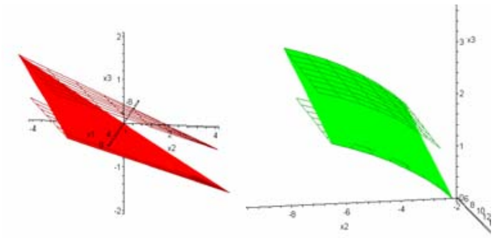

Figure 5: *PB* corresponding to the segments $RS^j{}_0$, $RS^j{}_1$

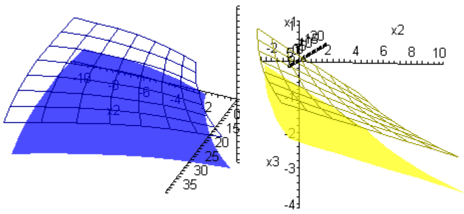

Figure 6: *PB* corresponding to the segments $RS^j{}_{12}$, $RS^j{}_2$

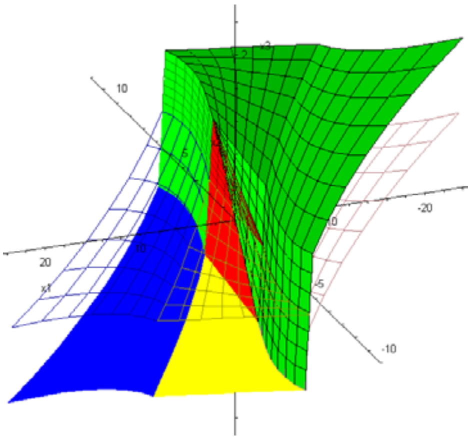

Figure 7: *RS* and *PB* in the phase space

4. NON-LINEAR CONTROL ALGORITHM

The control algorithm is based on achieving parameters $q_i, t_i, i = 1,2,3$. However, parameter t_3 is not needed, because there is enough to know whether $\mathbf{x}_3$ is in proportional band or saturation, so let us assign $t_3 = 0$. To find these parameters there is necessary to solve (15), however the results obtained by symbolic solutions can be used now. The formula for evaluation of $\mathbf{x}_3$ differs for each segment and it depends on $\mathbf{x}_1, \mathbf{x}_2$, so the control algorithm can be divided into following steps:

1. START
2. $RS^j{}_0$
   a. q_1, q_2, q_3 are unknown
   b. $t_1 = 0, t_2 = 0$
   c. solve (15)
   d. (10), (12) are not fulfilled GOTO 3
   e. $sat(q_3), u = \sum_{i=1}^{3} q_i$ GOTO 6
3. $RS^j{}_1$
   a. q_2, q_3, t_1 are unknown
   b. $q_1 = U_j, t_2 = 0$
   c. solve (15)
   d. $t_1 > 0$, (12) are not fulfilled GOTO 4
   e. $sat(q_3), u = \sum_{i=1}^{3} q_i$ GOTO 6
4. $RS^j{}_2$
   a. q_1, q_3, t_2 are unknown
   b. $q_2 = U_{3-j} - q_1, t_1 = 0$
   c. solve (15)
   d. $t_2 > 0$, (10) are not fulfilled GOTO 5
   e. $sat(q_3), u = \sum_{i=1}^{3} q_i$ GOTO 6
5. $RS^j{}_{12}$
   a. q_3, t_1, t_2 are unknown
   b. $q_1 = U_j, q_2 = U_{3-j} - U_j$
   c. solve (15)
   d. $t_1 > 0, t_2 > 0$ are not fulfilled GOTO ERROR
   e. $sat(q_3), u = \sum_{i=1}^{3} q_i$ GOTO 6
6. If the distance of the representative point from the desired state is greater than ε GOTO START
7. END

5. VERIFYING THE CONTROL ALGORITHM BY SIMULATION

The first simulation shows the three control phases presented in this paper.

Parameters of the simulation with initial state outside the *PB* are:

Chosen closed loop poles $\alpha_1 = -1.5, \alpha_2 = -3, \alpha_3 = -6$

Initial state $\mathbf{x} = [14.145, -6.097, 0.833]^T$

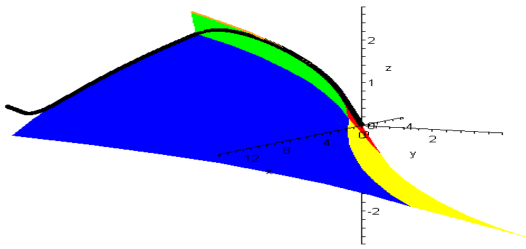

Figure 8: Trajectory in the phase space

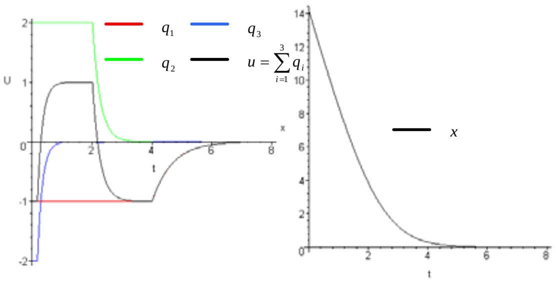

Figure 9: Control signal and output

Fig. 9 shows that the control signal approaches the constraint three times.

In the next simulation there is shown the trajectory starting at the x-axis. The symbolic solution of the algorithm for $RS^j{}_2$ is used in this case. Before, we had the symbolic solution only for the particular closed loop poles and for the particular constraints. The chosen closed loop poles in the simulation are $\alpha_1 = -1.5, \alpha_2 = -3, \alpha_3 = -6$ and the initial state is $\mathbf{x} = [30, 0, 0]^T$.

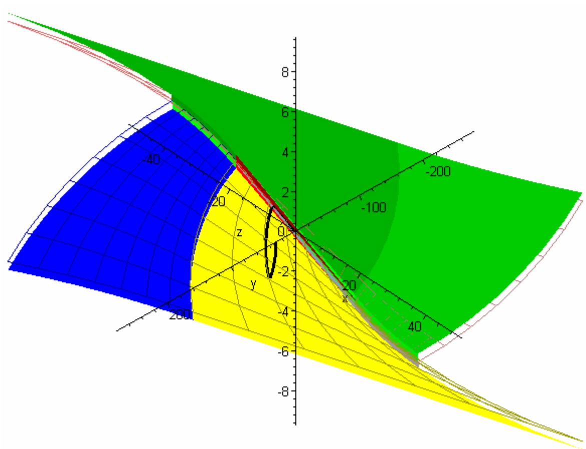

Figure 10: Trajectory in the phase space

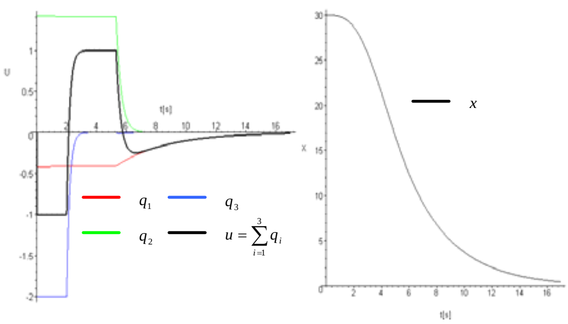

Figure 11: Control signal and output

6. CONCLUSION

The simulations have shown, that designed control consists of three phases well known from time optimal control, however the transients between these phases are "smooth" and they have dynamics given by the closed loop poles that gives the advantage to this algorithm in the field of nonlinear control. It can be used to control systems with parasitic transport delays etc. The interesting approach to the design of controller has shown that under the specified conditions (in this case given by input saturation limits) the symbolic computation are able to produce sophisticated solutions that can be finally evaluated by numerical calculations. Due to the symbolic solutions made in the MAPLE computer algebra system, the controller design process can be easily applied to different 3rd order systems.

ACKNOWLEDGMENTS

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