CONVEX EXTENSION OF DISCRETE-CONVEX FUNCTIONS AND APPLICATIONS IN OPTIMAL DESIGN OF COMPLICATED LOGISTIC, MANUFACTURING AND PROCESSING ENVIRONMENTS

Tiit Riismaa

Institute of Cybernetics at Tallinn University of Technology
Akadeemia tee 21, Tallinn, 12618

tiitr@ioc.ee

ABSTRACT
A method of description and optimal design of the structure of complicated multi-level processing systems is presented. The set of feasible structures for such class of systems is defined. The representation of this set is constructed in terms of the graph theory. For the reduced statement two types of variable parameters are defined: for the level size and for the relations of adjacent levels. The choice of variable parameters guarantees the discrete-convexity of objective function. A class of iteration methods for solving the discrete-convex programming problem is derived. The method based on the extension of discrete-convex function to the convex function and on extension of discrete-convex programming problem to the convex programming problem. On each step of the iteration the calculation of the value of objective function is required only on some vertices of unit cube. The considered approach is illustrated by an academic example of modelling and optimal design of the multi-level manufacturing system.

Keywords: discrete manufacturing and processing environment, optimal multi-level partitioning, discrete-convex function, nonlinear integer programming.

1. INTRODUCTION

Large-scale problems can be decomposed in many different ways (Mesarovic et al. 1970; Bruzzone et al. 2007). The current approach for describing and optimizing the structure of hierarchical systems is based on a multi-level partitioning of given finite set in which the qualities of the system may depend on the partitioning.

Examples of problems of this class are aggregation problems, structuring of decision-making systems, database structuring, the problems of multiple centralization or decentralization, multi-level selection problems, multi-level tournament systems (Laslier 1997), multi-level distribution systems, different clustering problems (Bruzzone et al. 2009).

In a multi-level distribution system each element is a supplier for some lower level elements and a customer for one higher-level element. The zero-level elements are only customers and the unique top-level element is only a supplier. The choice of optimal number of suppliers-customers on each level is a mathematically complicated problem.

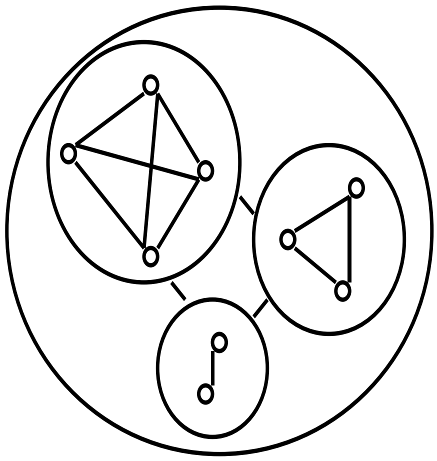

Figure 1: Multi-Level Partitioning of a Set of 9 Elements.

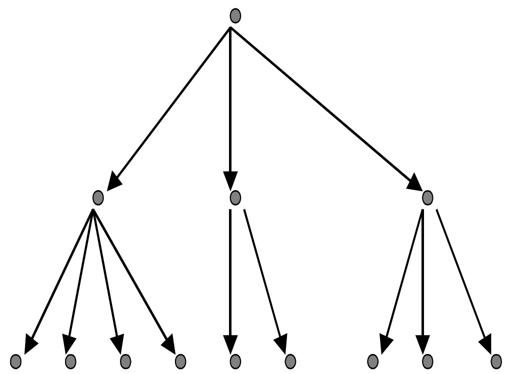

Figure 2: Hierarchy of Multi-Level Partitioning of a Set of 9 Elements

The tournament system (Laslier 1997) is a relatively simple special case of a multi-level processing system. To consider this system, the number of games (pair-wise comparisons) is a quadratic function of the number of participants. This is a quickly increasing function. If the number of participants is large, the number of games is very large. This is a reason why the multi-level approach is useful for the selection of the winner. From the tournaments of the first level the winners are distributed between the tournaments of the next level. The second level tournaments winners are going to the third level, until the winner is selected. Suppose the goal is to minimize the number of all games. If the price for all games is the same, the solution is well known. Each tournament has two participants and one game is played. If the prices of games for different levels are different or constraints to the number of levels are active, a relatively complicated nonlinear integer-programming problem arises.

Simulation model of logistic processes in container terminals allows considering terminal operation at three different partitioning levels (Merkuryev et al. 2003).

The main difficulty from the point of view of optimization is that the number of subsets of partitioning is a variable parameter. This means for corresponding optimization problem that upper limit of summation, the number of summands (integer valued parameter) is a variable parameter. It is hard to solve that kind discrete programming problem.

The advantage of the considered approach is that this choice of variables enables to extend the structure optimization problem to the convex programming problem. A finite steps algorithm converging to the global solution of this problem is presented.

2. FEASIBLE SET OF HIERARHIES

Consider all *s*-levels hierarchies, where nodes on level *i* are selected from the given nonempty and disjoint sets and all selected nodes are connected with selected nodes on adjacent levels. All oriented trees of this kind form the feasible set of hierarchies (Riismaa 1993). The illustration of this formalism is given in Figure 3.

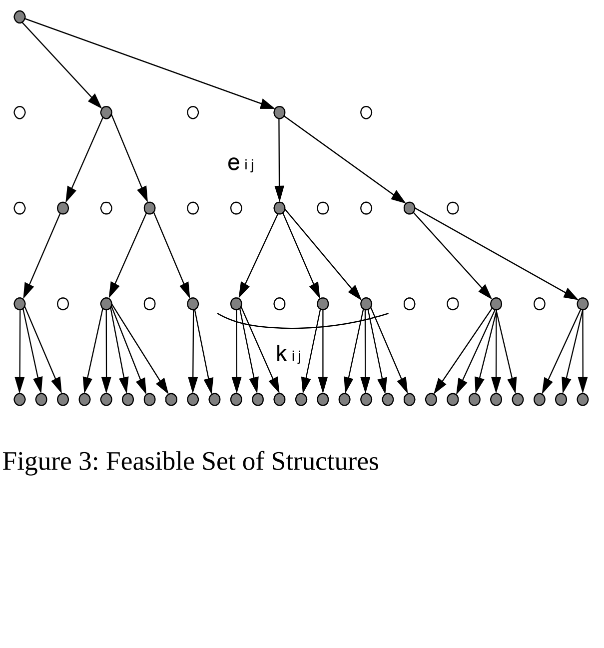

Figure 3: Feasible Set of Structures

Suppose $m_i \times m_{i-1}$ matrix $Y_i = \left(y^i_{jr}\right)$ is an adjacent matrix of levels *i* and $i-1$ $\left(i = 1,\dots,s\right)$ where

$$y^i_{jr} = \begin{cases} 1, \; j-th\ element\ on\ level\ i\ connected \\ \quad with\ r-th\ element\ on\ level\ i-1 \\ 0, otherwise \end{cases} .$$

Suppose m_0 is the number of 0-level elements (level of object, level of non-ordered set).

Theorem 1. All hierarchies with adjacent matrixes $\{Y_1,\dots,Y_s\}$ *from the described set of hierarchies satisfy the condition*

$$Y_s \cdot \ldots \cdot Y_1 = (\underbrace{1,\dots,1}_{m_0})$$

The assertion of this theorem is determined directly (Riismaa et al. 2003).

The illustration of multiplication of adjacency matrices is given on Figure 2. To the multiplication of adjacent matrices correspond the annihilation of levels. To the presentation an adjacent matrix as a product of two adjacent matrices correspond the creation a new level.

To the sequence of adjacent matrices $\{Y_1, Y_2, Y_3, Y_4\}$ corresponds the hierarchy where the arcs are described with continuous lines. To the sequence of adjacent matrices $\{Y_1, Y_3 \cdot Y_2, Y_4\}$ correspond the hierarchy where the arcs between the first and the second levels are described with dash lines and other arcs are described with continuous lines.

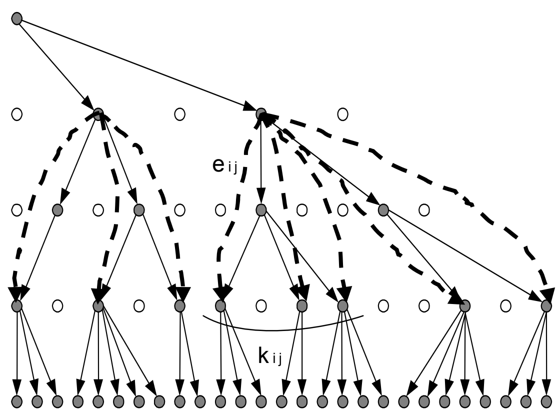

Figure 4: The Creation and annihilation of Levels

3. THE STATEMENT OF GENERAL PROBLEM OF STRUCTURE OPTIMIZATION

The general optimization problem is stated as a problem of selecting the feasible structure that corresponds to the minimum of total loss given in the separable-additive form:

$$\min_{Y_1,\ldots,Y_S}\left\{\sum_{i=1}^{s}\sum_{j=1}^{m_i}\mathrm{h}_{ij}\left(\sum_{r=1}^{m_{i-1}} d_{jr}^{\,i}\, y_{jr}^{\,i}\right)\middle|\begin{array}{l} Y_s\cdot\ldots\cdot Y_1 = \\ =\underbrace{(1,\ldots,1)}_{m_0}\end{array}\right\}. \quad (1)$$

Here $h_{ij}(\cdot)$ is an increasing loss function of *j*-th element on *i*-th level and $d_{jr}^{\,i}$ is the element of $m_i \times m_{i-1}$ matrix D_i for the cost of connection between the *i*-th and (*i*-1)-th level.

The meaning of functions $h_{ij}(k)$ depends on the type of the particular system.

4. REDUCED PROBLEM OF STRUCTURE OPTIMIZATON

Now an important special case is considered where the connection cost between the adjacent levels is the property of the supreme level: each row of the connection cost matrices between the adjacent levels consists of equal elements.

There is a possibility to change the variables and to represent the problem so that

$$d_{jr}^{\,i} = 1; \quad i = 1,\ldots s; \;\; j = 1,\ldots,m_i; \;\; r = 1,\ldots,m_{i-1}.$$

Now the total loss depends only on sums $\sum_{r=1}^{m_{i-1}} y_{jr}^{\,i} = k_{ij}$, where k_{ij} is the number of edges beginning in the *j*-th node on *i*-th level.

Recognize also that $\sum_{j=1}^{m_i} k_{ij} = p_{i-1}, \;\; i = 1,\ldots,s$, where p_i is the number of nodes on *i*-th level. If to suppose additionally that $h_{i1}(k) \le \cdots \le h_{im_i}(k)$ for each integer k, the general problem (1) transforms into the two mutually dependent phases:

$$\min_{p_1,\ldots,p_{S-1}}\left\{\sum_{i=1}^{s} g_i(p_{i-1},p_i)\,\middle|\; p_0 \ge \ldots \ge p_s,\;\; p_s = 1\right\} \quad (2)$$

where

$$g_i(p_{i-1},p_i) = \\ = \min_{k_{i1},\ldots,k_{ip_i}}\left\{\sum_{j=1}^{p_i} h_{ij}(k_{ij})\,\middle|\, \sum_{j=1}^{p_i} k_{ij} = p_{i-1}\right\} \quad (3)$$

Free variables of the inner minimization (3) are used to describe the connections between the adjacent levels. Free variables of the outer minimization (2) are used for the representation of the number of elements at each level.

5. CONVEX EXTENSION OF DISCRETE-CONVEX FUNCTIONS

This statement has some advantages from the point of view of the optimization technique. It is possible to adapt effective methods of the convex programming for solving outlined special cases.

The function $f: Z^n \to R$ is called discrete-convex (Riismaa 1993; Murota 2003) if for all $z_i \in Z^n (i = 1,\ldots,n+1)$, $Z^n = \underbrace{Z\times\ldots\times Z}_{n}$, $Z = \{\ldots,-2,-1,0,1,2,\ldots\}$, $\lambda_i \ge 0, (i = 1,\ldots,n+1)$ and

$$\sum_{i=1}^{n+1}\lambda_i = 1; \; \sum_{i=1}^{n+1}\lambda_i z_i \in Z^n$$

holds

$$f(\sum_{i=1}^{n+1}\lambda_i z_i) \le \sum_{i=1}^{n+1}\lambda_i f(z_i).$$

The use of all $n+1$ elements convex combinations follows from the well-known theorem of Caratheodory (Rockafellar 1970).

The graph of a discrete-convex function is a part of the graph of a convex function.

The convex extension f_c *of function* $f: X \to R\,(X \subset R^n)$ *is the majorant convex function* $f_c : conv\,X \to R$, *where* $f_c(x) = f(x)$ *if* $x \in X$.

Theorem 2. The function $f: X \to R$ $(X \subset R^n)$ *can be extended to convex function on* $convX$ *if* f *is discrete-convex on* X. *The convex extension* f_c *of* f *is*

$$f_c(x) = \min\left\{\sum_{i=1}^{n+1}\lambda_i f(x_i)\,\middle|\,\begin{array}{l} x = \sum_{i=1}^{n+1}\lambda_i x_i;\; \sum_{i=1}^{n+1}\lambda_i = 1; \\ \lambda_i \ge 0\;\; (i = 1,\ldots,n+1), \\ x_i \in X\;\; (i = 1,\ldots,n+1)\end{array}\right\}$$

over $x_i, \lambda_i\,(i = 1,...,n+1)$.

Assertion of this theorem is determined directly (Riismaa 1993).

Theorem 3. The convex extension f_C *of* f *is*

$$f_c(x) = \begin{cases} \max\limits_{a,b} \left\{ \langle a,x\rangle + b \,\middle|\, \begin{matrix} \langle a,y\rangle + b \le f(y), \\ y \in X \end{matrix} \right\}, \text{if} \\ x \notin X \\ f(x), \text{if } x \in X \end{cases}$$

Assertion is determined directly.

From theorem 2 or theorem 3, the convex extension is so called point-wise maximum over all linear functions not exceeding the given function.

From theorem 2 or theorem 3, the convex extension of a discrete-convex function is a piecewise linear function.

From theorem 2 and/or 3, each discrete-convex function has a unique convex extension.

From theorem 2 or theorem 3, the class of discrete-convex functions is the largest one to be extended to the convex functions.

Theorem 4. If $h_{ij}(k)(i = 1,...,s;\ j = 1,...,m_i)$ *in (3) are discrete-convex functions then* $\sum_{i=1}^{s} g_i(p_{i-1}, p_i)$ *in (2) is a discrete-convex function.*

The proof of this theorem is not very complicated but needs a lot of secondary results and can be found in (Riismaa 1993).

Considered theorem 2 or theorem 3 and theorem 4 enable to extend the objective function (2), (3) to the convex function.

6. ALGORITHM OF LOCAL SEARCHING FOR THE REDUCED PROBLEM OF STRUCTURE OPTIMIZATION

The particular choice of the variables (2), (3) enables to construct a class of methods for finding the global optimum. In this paper it is only declared that the objective function of such integer programming problem is a discrete-convex function.

Recall of (2)

$g(p_0, p_1,..., p_{s-1},1) = \sum_{i=1}^{s} g_i(p_{i-1}, p_i)$ and denote $p_k^{(s)} = \left(p_{k1}^{(s)},..., p_{ks-1}^{(s)},1\right)\ (k = 0,1,...)$.

Consider following finite-step algorithm:

$p_0^{(s)} = (\underbrace{1,...,1}_{s})$, and $p_k^{(s)} = p_{k-1}^{(s)} + x_k^{(s)}(q,t)$, $(k = 1,2,...)$.

It is assumed that

1) $x_k^{(s)}(q,t)$ are lexicographically ordered by (q,t):

$$x_k^{(s)}(q,t) = \left(\overbrace{\left(\underbrace{0,...,0}_{t}\right), \left(\underbrace{1,...,1}_{q}\right), 0,...,0}^{s} \right),$$

$$(q = 1,...,s-t-1;\ t = 0,...,s-2); \tag{4}$$

2)

$$p_0 \ge p_{k1}^{(s)} \ge ... \ge p_{ks}^{(s)} = 1 \tag{5}$$

3) $x_k^{(s)}(q,t)$ is lexicographically the first that satisfies the condition

$$\begin{aligned} & g\left(p_0, p_{k1}^{(s)},..., p_{ks-1}^{(s)},1\right) \le \\ & \le g\left(p_0, p_{k-11}^{(s)},..., p_{k-1s-1}^{(s)},1\right) \end{aligned} \tag{6}$$

Remark 1. Consider the vertices with integer valued coordinates of the s-dimensional unit cube where:

- the nearest vertex to the s-dimensional zero-point is $p_{k-1}^{(s)} = \left(p_{k-11}^{(s)},..., p_{k-1s-1}^{(s)},1\right)$;
- other vertices $p_{k-1}^{(s)} + x_k^{(s)}(q,t)$ satisfy the condition (4).

The number of that kind of vertices (4) is $\frac{1}{2} \cdot s(s-1)$. The number of vertices (4), (5) is no more than $\frac{1}{2} \cdot s(s-1)$.

The condition (4) puts in order all vertices of described unit cube.

Remark 2. On the iteration step *k* the value of goal function is computed on ordered vertices (4), (5) of unit cube until the first value satisfying (6) is found. If that kind of a value does not exist, one of the solutions of problem (2) - (3) has been found.

7. ACADEMIC EXAMPLE: OPTIMIZATION THE STRUCTURE OF COMPLICATED MULTI-LEVEL MANUFACTURING SYSTEM

Consider the processing of *n* parts (Riismaa et al. 2003). In case of one processing unit the overall processing and waiting time for all n parts is proportional to n^2 and is a quickly increasing function. For this reason the hierarchical system of processing can be suitable. From zero-level (level of object) the parts will be distributed between p_1 first-level processing units and processed (aggregated, packed etc.) by these units. After that the parts will be distributed between p_2 second-level processing units and processed further and so on. From p_{s-1} (s - 1)-level the units will be sent to the unique s-level unit and processed finally. The cost of processing and waiting on level i is approximately

$$g_i(p_{i-1}, p_i) = (d_i\, l_{i-1} p_{i-1} / p_i)^2\, p_i + a_i\, p_i$$
$(i = 1,..., s)$.

Here l_i is the number of aggregates produced by one robot on level i (a number of boxes for packing unit), d_i is a loss unit inside the level i, and a_i is the cost of i-th level processing unit. The variable parameters are the number of processing units on each level $p_i(i = 1,..., s)$.

The goal is to minimize the total loss (processing time, waiting time, the cost of processing units) over all levels:

$$\min \sum_{i=1}^{s} \left((d_i l_{i-1})^2 \begin{pmatrix} \left(p_i \left(\left[\frac{p_{i-1}}{p_i} \right] + 1 \right) - p_{i-1} \right) \left[\frac{p_{i-1}}{p_i} \right]^2 + \\ + \left(p_{i-1} - p_i \left[\frac{p_{i-1}}{p_i} \right] \right) \left(\left[\frac{p_{i-1}}{p_i} \right] + 1 \right)^2 \end{pmatrix} + a_i p_i \right)$$

over natural $p_i(i = 1,..., s)$.

Here $[p]$ is the integer part of p. The goal function of this discrete programming problem is discrete-convex. It is possible to extend this function to convex function (Theorem 2) and get a solvable convex programming problem using the method of local searching.

8. CONCLUSION

Many finite hierarchical structuring problems can be formulated mathematically as a multi-level partitioning procedure of a finite set of nonempty subsets. This partitioning procedure is considered as a hierarchy where to the subsets of partitioning correspond nodes of hierarchy and the relation of containing of subsets define the arcs of the hierarchy. The feasible set of structures is a set of hierarchies (oriented trees) corresponding to the full set of multi-level partitioning of given finite set.

Each tree from this set is represented by a sequence of Boolean matrices, where each of these matrices is an adjacency matrix of neighboring levels. To guarantee the feasibility of the representation, the sequence of Boolean matrices must satisfy some conditions – a set of linear and nonlinear equalities and inequalities.

The formalism described in this paper enables to state the reduced problem as a two-phase mutually dependent discrete optimization problem and construct some classes of solution methods. Variable parameters of the inner minimization problem are used for the description of connections between adjacent levels. Variable parameters of the outer minimization problem are used for the presentation of the number of elements on each level.

The two-phase statement of optimization problem guarantees the possibility to extend the objective function to the convex function and enables to construct algorithms for finding the global optimum. In this paper for finding the global optimum the method of local searching is constructed. On each step of iteration the calculation of the value of objective function is required only on some vertices of some kind of unit cube.

REFERENCES

Bruzzone, A. G., Bocca, E., Pierfederici, L., 2009. Multilevel forecasting models in manufacturing systems. *Proceedings of the 2009 Spring Simulation Multiconference*. March 22 – 27, San Diego (California, USA).

Bruzzone, A. G., Briano, E., Bocca, E., Massei, M., 2007. Evaluation of the impact of different human factor models on industrial and business processes. *Simulation Modelling Practice and Theory*, 15 (2), 199 – 218.

Laslier, J.-F., 1997. *Tournament solutions and majority voting*. Berlin: Springer.

Merkuryev, Y., Bardatchenko, V., Solomennikov, A. and Kamperman, F. Simulation of logic process at the Baltic container terminal: model validation and application. *Proceedings of the International Workshop on Harbour, Maritime and Multimodal Logistics Modelling & Simulation*, pp. 294 – 302, September 18 – 20, 2003, Riga, Latvia.

Mesarovic, M. D., Macko, D. and Takahara, Y., 1970. *Theory of hierarchical multi-level systems*. New York: Academic Press.

Murota, K., 2003. *Discrete convex analysis*. Philadelphia: SIAM.

Riismaa, T., 1993. Description and optimization of hierarchical systems. *Automation and Remote Control* 12, 1852 - 1857.

Riismaa, T., Randvee, I., Vain, J., 2003. Optimization of the Structure of Multi-Level Parallel Assembling. *Proceedings of 7th IFAC Workshop Intelligent manufacturing Systems*, pp. 235 – 238, April 6 – 8, 2003, Budapest (Hungary).

Rockafellar R. T., 1970. *Convex analysis*. New Jersey: Princeton University Press.

AUTHORS BIOGRAPHY

Tiit Riismaa graduated from the Tartu University in 1972 as a mathematician and received a Ph.D. in 1986 in mathematical cybernetics at the Computing Centre of Russian Academy of Science. Now he is the researcher for the Institute of Cybernetics at Tallinn University of Technology. His fields of research include mathematical theory of systems, hierarchical systems, optimization of structures, mathematical programming.