THE STUDY OF A DETERIORATING MANUFACTURING SYSTEM USING SIMULATION AND RESPONSE METHODOLOGY

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ABSTRACT
A deteriorating production system consisting of two parallel machines with the production dependent failure rates of the machine is investigated in this paper. The machines produce one type of final products. The demand rate for the final commodity is constant and unmet demand is backlogged. The goal of the control problem is to find the production rates of both machines so as to minimize a long term average expected cost which penalizes both the presence of waiting customers and the inventory. In the proposed model, the production rate of the first machine is higher than the production rate of the second machine. The failure rate of the first machine which is the main machine depends on its production rate. The failure rate of the second machine is constant. The proposed model is based on a Markov decision process, and the stochastic dynamic programming method is used to obtain the optimality conditions. Control policy parameters are obtained by combining analytical modelling, simulation experiments and response surface methodology. Sensitivity analyses of the optimal results with respect to the system parameters are also examined to illustrate the importance and effectiveness of the proposed methodology. The usefulness of the proposed approach is outlined for more complex situations where the system must deal with non-exponential failure and multiple machines.

Keywords: production planning, stochastic dynamic programming, numerical methods, simulation

1. INTRODUCTION
Due to the constant search for increased productivity, a better service to clients, the number of scientific publications in the field of failure prone manufacturing systems has been steadily growing.

This paper investigates a stochastic deteriorating production system consisting of two parallel machines with the production rate-dependent failure rates of the machine. The stochastic nature of the system is due to machines that are subject to random breakdowns and repairs. The machines produce one part type; whenever a breakdown occurs, a corrective maintenance is performed to restore the machines to its operational mode. Our objective is to find the production rates of the different machines so as to minimize a long term average expected cost including inventory and backlog costs. To solve the optimization problem of this paper, we propose a stochastic programming formulation of the problem and derive the optimal production policies numerically. Control policy parameters are obtained combining analytical modelling, simulation experiments and response surface methodology.

An overview of relevant literature reveals that significant contributions have been proposed based on: two parallel machines manufacturing systems (Sajadi et al. 2011), one machine with the failure rate depends on the production rate (Martinelli 2010) and a combination of the control theory and the simulation-based experimental design (Gharbi et al. 2011). This paper's main contribution lies in the study of a stochastic manufacturing system consisting of two parallel machines with the production dependent failure rates of the main machine.

A common feature of this paper is that the policies are of the hedging point type and depend on multiple thresholds. The methodology presented in this paper can be applied in the machining mechanical parts industry where there are many different parallel machines. Some of them are classical machines (constant failure rates) and the others are degraded (if they work at faster rates, they are more likely to fail).

2. STATEMENT OF THE PROBLEM
As illustrated in Figure 1, the manufacturing system studied consists of two parallel machines producing one part type denoted M_1 and M_2. The machines are subject to random breakdowns and repairs. The repair rate is constant. The maximum production rates of machines are known and the demand for finished products is deterministic. The failure rate of M_1 which is the main machine (machine whose production rate is the highest) depends on its production rate. Then, when this machine

works at a faster rate, it is more likely to fail. In contrast, the failure rate of the second machine is constant. We assumed that M_2 can't meet the customer demand alone. The stochastic nature of the system is related to breakdowns and repairs of machines.
The state of the machines can be classified as:

- state 1 (mode 1): M_1 and M_2 are operational
- state 2: M_1 is operational and M_2 is under repair
- state 3: M_1 is under repair and M_2 is operational
- state 4: M_1 and M_2 are under repair.

We use $\xi(t)$ to denote the state of the machines with value in $B=\{1,2,3,4\}$. The dynamic of the system is described by a discrete element $\xi(t)$ and a continuous element $x(t)$. The discrete element represents the status of the machines and the continuous one, the stock level. It can be positive for an inventory or negative for a backlog.

The discrete part of the system is a continuous time Markov process, with a transition rate from state α to state β denoted by $q^{\theta}_{\alpha\beta}$ with $\alpha,\beta \in B$. For the considered system, the corresponding 4×4 transition matrix $Q=\left[q^{\theta}_{\alpha\beta}\right]$ is one of an ergodic process as defined in Ross (2003).

We assume that the failure rate of the 1st machine depends on its production rate and is defined by:

$$\begin{array}{ll} q_{12}^{11} & if\ u_1 \in \left(U, u_{1\max}\right] \\ q_{12}^{12} & if\ u_1 \in \left[0, U\right] \end{array} \tag{1}$$

with $q_{12}^{11} \geq q_{12}^{12} \geq 0$ and $0 \leq U \leq u_{1\max}$

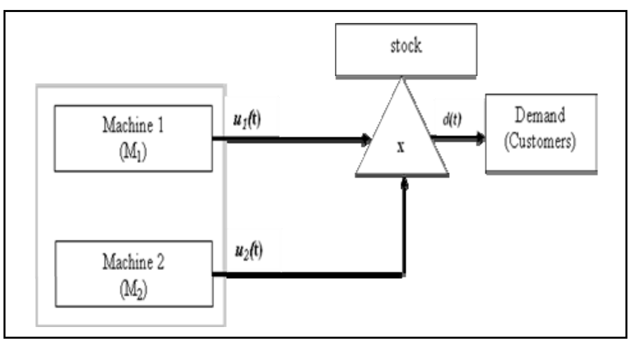


Figure 1: System under study

The transition rates verify the following conditions:

$$q^{\theta}_{\alpha\beta} \geq 0 \quad (\alpha \neq \beta) \tag{2}$$

$$q^{\theta}_{\alpha\alpha} = -\sum_{\beta\neq\alpha} q^{\theta}_{\alpha\beta} \tag{3}$$

The transition probabilities are given by:

$$\mathrm{P}\left[\xi(t+\delta t)=\beta \middle| \xi(t)=\alpha\right] = \begin{cases} q^{\theta}_{\alpha\beta}(.)\,\delta t + o(\delta t) & if\ \alpha\neq\beta \\ 1+q^{\theta}_{\alpha\beta}(.)\,\delta t + o(\delta t) & if\ \alpha=\beta \end{cases} \tag{4}$$

with $\lim\limits_{\delta t\to 0} \dfrac{o(\delta t)}{\delta t} = 0$ for all $\alpha,\beta \in B$.

Let $u_1(t)$ and $u_2(t)$ denote the production rates of M_1 and M_2 respectively, in mode α and at time t. The set of the feasible control policies $\mathrm{A}(\alpha)$, including $u_1(\cdot)$ and $u_2(\cdot)$ depends on the stochastic process $\xi(t)$ and is given by:

$$\mathrm{A}(\alpha) = \left\{ \begin{array}{l} \left(u_1(\cdot), u_2(\cdot)\right) \in \Re^2,\ 0 \leq u_1(\alpha,\cdot) \leq u_{1\max}, \\ 0 \leq u_2(\alpha,\cdot) \leq u_{2\max} \end{array} \right\} \tag{5}$$

where $u_1(\cdot)$ and $u_2(\cdot)$ are known as control variables, and constitute the control policies of the problem under study.

The continuous part of the system dynamics is described by the following differential equation:

$$\frac{dx(t)}{dt} = u_1(t) + u_2(t) - d,\ x(0) = x \tag{6}$$

Let $g(\cdot)$ be the cost rate defined as follows:

$$g(\alpha, x, \cdot) = c^{+}x^{+} + c^{-}x^{-} \tag{7}$$

The constants c^{+} and c^{-} ($ per parts per unit of time) are used to penalize inventory and backlog respectively,

$x^{+} = \max(0, x),\ x^{-} = \max(-x, 0)$

The problem here is to control the production rates of the both machines. The performance criterion considered is the expected discounted cost $J(\cdot)$ given by:

$$J(\alpha, x, u_1, u_2) = E\left\{ \begin{array}{l} \int_0^{\infty} e^{-\rho t} g(\alpha, x, \cdot)dt \\ \left| x(0)=x, \xi(0)=\alpha \right. \end{array} \right\} \tag{8}$$

where ρ is the discount rate. The value function of such a problem is defined as follows:

$$v(\alpha, x) = \inf_{(u_1(\cdot), u_2(\cdot))\in \mathrm{A}(\alpha)} J(\alpha, x, u_1, u_2)\ \ \forall \alpha \in B \tag{9}$$

In Appendix A, we present the optimality conditions and the numerical methods used to solve them for the value function $v(\cdot)$ given by equation (9). The contribution of this research to the Hamilton-Jacobi-Bellman (HJB) equations is that in the modes 1 and 2 where M_1 is operational, we have four equations instead of two equations in the case of a manufacturing system without production rate dependent failure rate (see equations A.3 and A.4). The next section provides a numerical example to illustrate the structure of the control policies.

3. NUMERICAL RESULTS AND SENSITIVITY ANALYSES

3.1. Numerical results

In this section, we present a numerical example for the manufacturing system presented in Section 2. A four-state Markov process with the modes in $B = \{1, 2, 3, 4\}$ describes the system capacity. The instantaneous cost is described by equation (7).

The considered computation domain D is given by:

$$D = \{x : -20 \le x \le 40\} \tag{10}$$

The condition to meet the customer demands, over an infinite horizon and reach a steady state is given by:

$$\begin{cases} \pi_1 \cdot (u_{1\max} + u_{2\max}) + \pi_2 \cdot u_{1\max} + \pi_3 \cdot u_{2\max} > d \\ \pi_1 \cdot (U + u_{2\max}) + \pi_2 \cdot U + \pi_3 \cdot u_{2\max} > d \end{cases} \tag{11}$$

Where (π_1, π_2 and π_3) are the limiting probability at the operational modes of the machines. Note that the limiting probabilities of modes 1, 2, 3 and 4 (i.e., π_1, π_2, π_3 and π_4), are computed as follows:

$$\pi \cdot Q(\cdot) = 0 \quad \text{and} \quad \sum_{i=1}^{4} \pi_i = 1 \tag{12}$$

where $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ and $Q(\cdot)$ is the corresponding 4×4 generator matrix. Table 1 summarizes the parameters of the numerical example for which the feasibility conditions given by equation (11) are satisfied.

Table 1: Parameters of numerical example

c^+	c^-	h	U	$u_{1\max}$	$u_{2\max}$	d_1	d_2	d	q_{12}^{11}	q_{12}^{12}	q_{12}^{2}	q_{21}^{1}	q_{21}^{2}	ρ	q_{12}^{11}
1	100	0.5	0.70	1.2	0.65	0.5	0.5	1	0.03	0.02	0.04	0.1	0.2	0.5	0.3

Figures 2 and 3 represent the production rates at mode 1 of machines M_1 and M_2 respectively. In these figures, we can see that the thresholds z_1 and z_3 are low because both machines are operational. The results of Figure 3 suggest that as the inventory level approaches a hedging point level, it may be beneficial to decrease the production rate to gain in reliability. Figures 2 and 3 show that the production rates are set to zero for comfortable stock levels. Then, there is no need to produce parts for comfortable stock levels. From the results obtained, the computational domain of Figure 3 (M_2) is divided into three regions as in Akella and Kumar (1986) and references therein. However, the computational domain of Figure 2 is divided into four regions. This is the main contribution of this paper. The optimal production control policy consists of one of the following rules:

1. Set the production rate of M_1 to its maximal value when the current stock level is under the first threshold value ($z_1 = 0.0$);
2. Reduce the production rate of M_1 to its minimal value when the current stock level approaches the second threshold value ($z_2 = 13.0$);
3. Set the production rate of M_1 to the demand rate when the current stock level is equal to the second threshold value;
4. Set the production rate of M_1 to zero when the current stock level is larger than the second threshold value.

The control policies obtained are the multi-hedging point policies. As shown within the numerical results and in Figure 2 and 3, the optimal production rates can be expressed as follows:

$$u_1(x,1) = \begin{cases} u_{1\max} & \text{if } x < z_1 = 0.0 \\ U & \text{if } z_1 \le x < z_2 = 13.0 \\ d_1 & \text{if } x = z_2 \\ 0 & \text{if } x > z_2 \end{cases} \tag{13}$$

where z_1 and z_2 are the first and second optimal threshold values of M_1 respectively.

$$u_2(x,1) = \begin{cases} u_{2\max} & \text{if } x < z_3 = 1.5 \\ d_2 & \text{if } x = z_3 \\ 0 & \text{if } x > z_3 \end{cases} \tag{14}$$

where z_3 is the optimal threshold value of M_2.

Unlike in Figure 2 where the tendency was to use less the maximal production rate of the first machine, Figure 4 shows that the first threshold ($z_4 = 1.0$) is higher than the case of Figure 2 because the machine works alone. However, the control policy is still a multi-hedging point policy and is defined by:

$$u_1(x,2) = \begin{cases} u_{1\max} & \text{if } x < z_4 = 1.0 \\ U & \text{if } z_4 \le x < z_5 = 7.5 \\ d_1 & \text{if } x = z_5 \\ 0 & \text{if } x > z_5 \end{cases} \tag{15}$$

where z_4 and z_5 are the first and second optimal threshold values of M_1 respectively.

Using the control policies given by equations (13), (14) and (15), the company will be able to take into account the availability of machines. Then, it can minimize the total cost due to failure of machines, allowing it to eventually maximize its total profit. The next section analyses the sensitivity of the policies obtained and several experimentations are conducted to ensure that the structure of the obtained policy is maintained and can be considered as a generalized policy for the general problem under study.

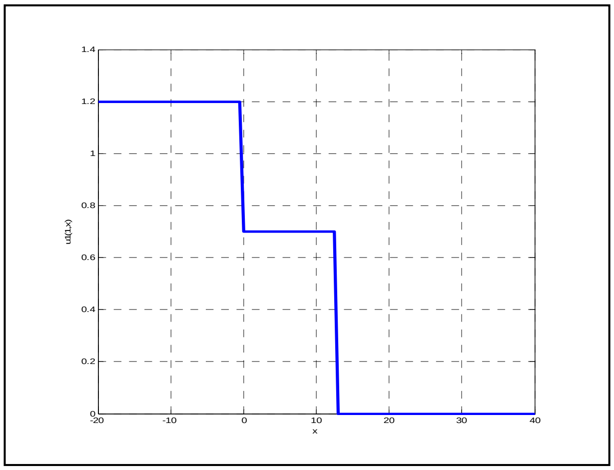


Figure 2: Production rate of M_1 at mode 1

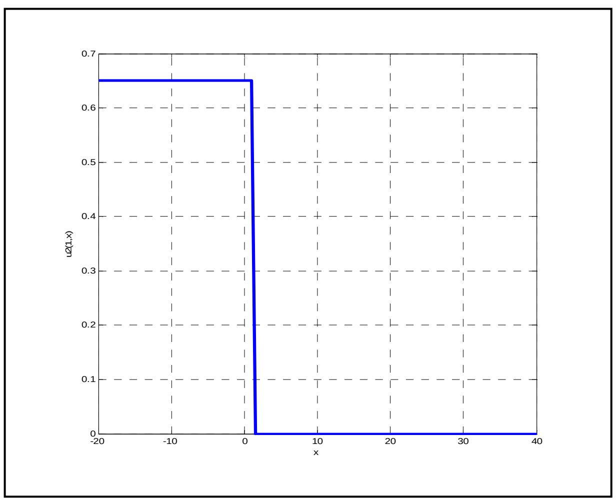


Figure 3: Production rate of M_2 at mode 1

3.2. Sensitivity analyses

A set of numerical examples are considered to measure the sensitivity of the obtained control policies and to illustrate the contribution of this paper. The sensitivity of the control policies is analyzed according to the variation of the backlog costs.

The results presented in Figures 5 and 6 show the behavior of the production rates of machines according to the variation of backlog costs. Based on these results, we can see that the value of the backlog costs is not too much impact the threshold z_1. This is logical because at mode 1, when both machines are operational, it was less use a first machine to its maximal production rate to take into account its reliability. The thresholds z_2 and z_3 increase in order to avoid further backlog costs. However, z_3 is far less than z_2. Thus, it does not use the second machine a lot when both machines are producing. One prefers to use M_1 to its minimal production rate because the failure rate depends on the rate of production (for a low production rate, the probability to fail is low).

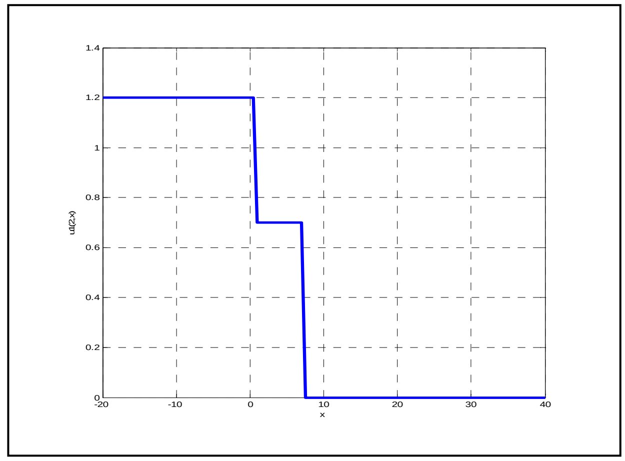


Figure 4: Production rate of M_1 at mode 2

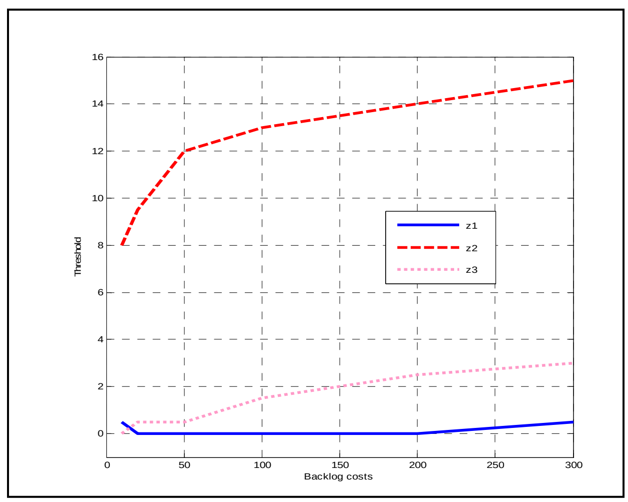


Figure 5: Threshold value at mode 1 versus backlog costs

Figure 6 shows that the threshold values of M_1 at mode 2 increase as the backlog costs increase. We therefore need a lot of parts in stock to avoid further backlog costs.

Through the observations made from the sensitivity analysis, it clearly appears that the results obtained make sense and confirm and validate the proposed approach. It shows the usefulness of the proposed model given that the control policies move as predicted, from a practical view point.

4. PROPOSED SIMULATION BASED OPTIMAL APPROACH

The results from traditional methods of planning in the environment of manufacturing systems are not sufficient to reach a comfortable level of desired performances. To improve these methods, a combination of the control theory and the simulation-based experimental design, as in Gharbi et al. (2011), is used to obtain a near-optimal control policy. This could allow the possibility of developing more realistic cases. To quantity the policy, which structure is given by analytical model, simulation model are combined with experimental design and response surface methodology to estimate the optimal values of the policy's parameters. In the case of non-exponential failure distribution, the quantification parameters are also possible with the help of the simulation model, which can easily take into account the nature of any probability distributions. The incurred cost is then given by simulation model which affects the response surface model.

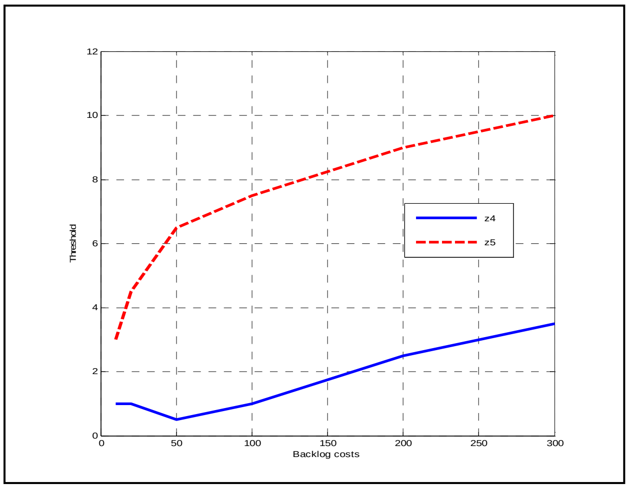


Figure 6: Threshold value at mode 2 versus backlog costs

5. SIMULATION MODEL

A discrete event simulation model that described the dynamics of the system is developed using Arena software (Arena is a powerful modeling and simulation software tool that allows the user to construct a simulation model run experiments. It generates several reports as a result of a simulation run). In order to obtain the cost of the system for a given set of input factors, the behavior of the system is simulated following the diagram shown in Figure 7 with the following block descriptions:

1. The *initialization* block sets the values of threshold (z_1,z_2 and z_3), the demand rate (d), and the machines parameters ($U, u_{1\max}, u_{2\max}, q_{12}^{11}, q_{12}^{12}, q_{12}^{2}$, q_{21}^{1} and q_{21}^{2}), etc. The simulation time T_{sim} is also assigned at this step.
2. The *arrival demand* block performs the arrival of the demand for the production system at each $1/d$ unit of time. Verification is then performed on the inventory values. The inventory or the backlog level is then updated.

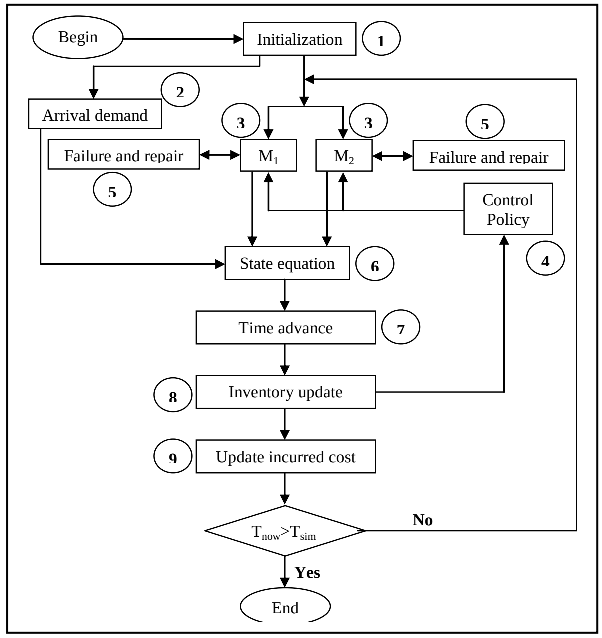


Figure 7: Diagram of simulation model

3. The *M_1 and M_2* blocks represent the main machine and the second machine respectively. The machines are subject to random failures and repairs.
4. The *control policy* block is defined in Section 3 (Equations 13-15) for the system production rates. The control policy is given by the output of the inventory update block. This block permanently sends signals to verify the variation in the stock level $x(t)$.
5. The failure and repair blocks sample the times to failure (MTBF1and $MTBF_2$ of M_1, and $MTBF_3$ of M_2) and time to repair $MTTR_1$ ($(q_{21}^{1})^{-1}$) and $MTTR_2$ ($(q_{21}^{2})^{-1}$) of the first machine and the second machine respectively.
6. The *state equation* is given by (5). It describes the inventory and backlog variables using the production rates set by the control policy and the variables from the failures and repairs of machines M_1 and M_2.
7. The *time advance* block uses an algorithm provided by simulation software. It is a combination of discrete event scheduling (failures and repairs), continuous variable threshold crossing events and time step specifications.
8. The *inventory update* block updates inventories when a unit is produced or when a unit of demand for the final product occurs.

9. The *update occurred cost* block calculates the average total costs according to the levels of the inventory and backlog variables (x^+ and x^-), their corresponding costs (c^+ and c^-) and which machine is producing (M_1 and/or M_2).

The simulation runs until the current time T_{now} reaches the simulation horizon T_{sim}, which is the time needed to reach the steady state. We perform five replications of the simulation model.

6. EXPERIMENTAL DESIGN AND RESPONSE SURFACE METHODOLOGY

Given that an optimal solution of the stochastic control problem described in Section 2 exists and given the convexity property of the cost function, we define three levels for each factor to obtain a convex estimated cost function. For these raisons, a complete 3^3 experimental design and a second-order response surface model were proposed.

6.1. Numerical example

For the numerical example experiment in this section, the following values are used:

$d = 1$ units/UT, $u_{1\max} = 1.2$ units/UT, U = 0.7 units/UT, $u_{2\max} = 0.65$ units/UT, $(q_{12}^{11})^{-1} = 33$ UT, $(q_{12}^{12})^{-1} = 50$ UT, $(q_{12}^{2})^{-1} = 25$ UT, $(q_{21}^{1})^{-1} = 10$ UT, $(q_{21}^{2})^{-1} = 5$ UT, $c^+ = 10$ \$/unit/UT, $c^- = 100$ \$/unit/UT.

We also defined a new variable $a = \dfrac{z_1}{z_2}$ with $0 \le a \le 1$ to ensure that the constraint $z_1 < z_2$ is respected. The minimum and maximum values of z_2 and z_3 were first observed using simulation experiments. The independent variable levels were then chosen as presented in Table 2.

Table 2: Level of independent variables

Factors	Low level	High level
a	0	1
z_2	0	20
z_3	0	20

We selected a 3^3 response surface design since we have three independent variables at three levels each. This design leads to the completion of $81\ (3^3 \times 3)$ experimental trials. To ensure that the steady state of the cost was achieved, the simulation model was run during 25 000 months for each replication (the simulation was run for 5 replications).

6.2. Results analysis

The statistical analysis of the simulated data consists of the multi-factor analysis of variance (ANOVA). This is done using a statistical software application (STATGRAPHICS) to provide the effects of the three independent variables (z_1, z_2 and z_3) on the dependent variable (Total cost). The ANOVA table for this model is summarized in Table 3. For each main effect, interaction and quadratic effect, Table 3 includes the sum of squares, the degree of freedom (df), the mean square, an F-ratio, computed using the residual mean square, and the significance level of the P-value. The factors, the quadratics effects and the interactions were considered significant at p-values less than 5% ($p < 0.05$). The $R^2_{adjusted}$ value of 0.9231 from the ANOVA table states that more than 92% of the total variability is explained by the model (Montgomery 2005).

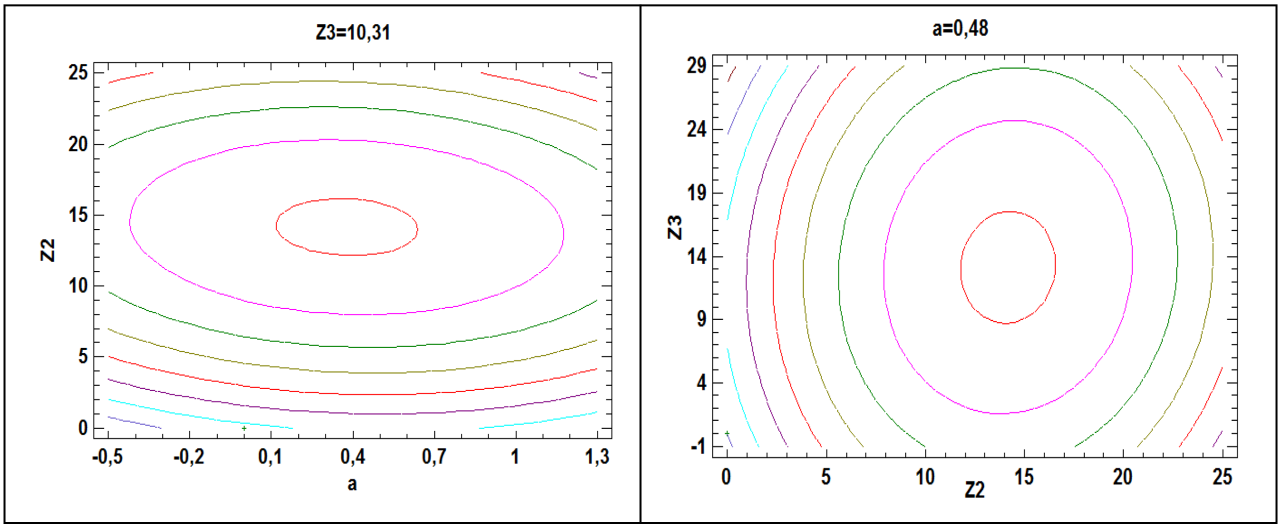
Table 3: ANOVA table

	Sum of squares	d.f	Mean square	F-ratio	P-value
a	1117,93	1	1117,93	1,28	0,2625
z_2	504600,	1	504600,	576,09	0,0000
z_3	17077,3	1	17077,3	19,50	0,0000
aa	5586,24	1	5586,24	6,38	0,0139
az_2	1840,41	1	1840,41	2,10	0,1517
az_3	43597,4	1	43597,4	49,77	0,0000
z_2z_2	254185,	1	254185,	290,19	0,0000
z_2z_3	1592,01	1	1592,01	1,82	0,1820
z_3z_3	22071,0	1	22071,0	25,20	0,0000
Total error	60437,9	69	875,912		
Total (corr.)	912105,	80			
$R^2_{adjusted} = 92,32\%$ $(z_1 = a * z_2)$					

The residual analysis was used to verify the adequacy of the model. A residual versus predicted value plot and normal probability plot were used to test the homogeneity of the variances and the residual normality, respectively. It can be concluded that the model is satisfactory. Due to the convexity property of the value function, the second-order response surface method was selected. The third-order interactions and all other effects were ignored. The estimated second-order model of the total cost is given by:

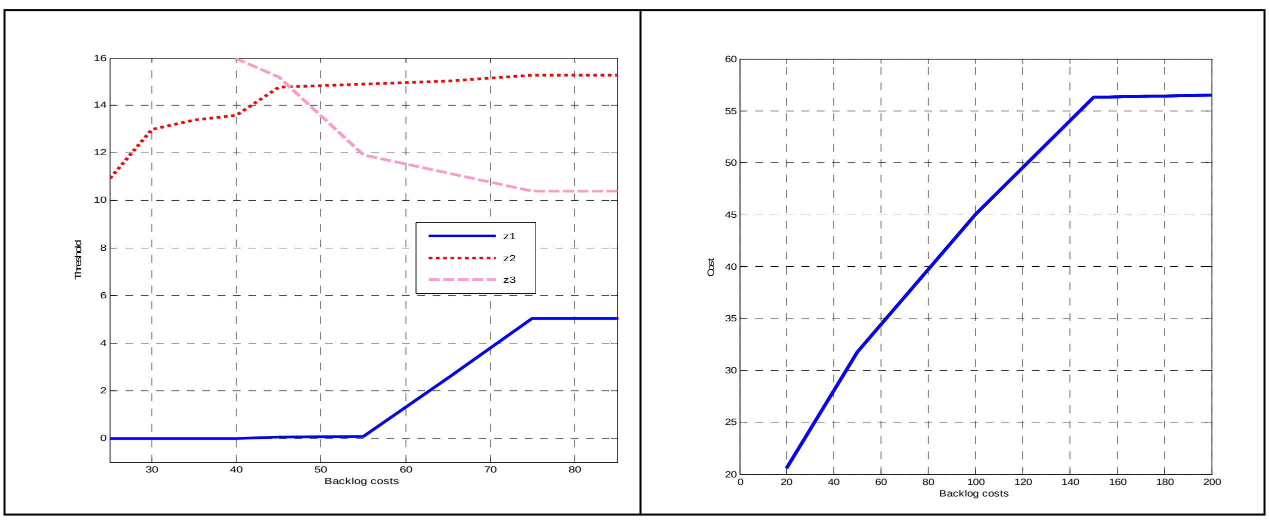
$$J = 5373{,}144 - 145{,}567 \cdot a - 33{,}4833 \cdot z_2 - 11{,}5967 \cdot z_3 + 70{,}4667 \cdot a^2 + 1{,}43 \cdot a \cdot z_2 + 6{,}96 \cdot a \cdot z_3 + 1{,}18833 \cdot z_2^2 - 0{,}0665 \cdot z_2 \cdot z_3 + 0{,}350167 \cdot z_3^2 \quad (16)$$

The projection of the corresponding cost response surfaces onto two-dimensional planes are presented in Figures 8(a) and 8(b). The minimum of the cost function, $J^* = 45{,}03$ is located at $z_2^* = 11{,}31$, $a^* = 0{,}477$ ($z_1^* = 5{,}39$), $z_3^* = 10{,}31$. These values define the best values to be applied to the manufacturing system considered.



(a) (b)

Figure 8: Contour plot of the response surface



(a) Threshold values (b) Total cost

Figure 9: Trend of threshold values and total cost versus backlog costs

6.3. Sensitivity analysis

Another set of experiments is considered to measure the sensitivity of the obtained control policy with respect to backlog costs (i.e. c^-). The following variations, illustrated in Figure 9 ((a) and (b)) are explored and compared to the basic case ($c^- = 100$).

The results show that when the backlog costs decrease, the threshold levels of M_1 (z_1^* and z_2^*) decrease in order to avoid further inventory costs, z_3^* increases. Consequently, when both machines are operational, the first machine has to work less to take into account its reliability. The overall cost decrease. Increasing c^- results in a tendency to increase the threshold values z_1^* and z_2^* in order to avoid further backlog costs. The total cost also increases and the threshold z_3^*

decreases. Thus, it does not use the second machine a lot when both machines are producing.

It clearly appears that the results obtained and discussed are coherent and confirm the numerical observation in the sense that when a cost decreases (resp. increases), the area where this costs is incurred increases (resp. decreases). But, in the case of the second machine, the chart of z_3 gives the opposite results compared with the numerical method. The simulation-based experimental design suggests using the main machine a lot when the backlog costs increase in order to avoid further backlog costs. We recall that the failure rates of the main machine depend on its production rate. Then, we have the possibility to act on its production rate.

7. CONCLUSION

Hedging point and multiple thresholds hedging point are piecewise constant control policies that can be easily implemented for planning of non-homogeneous Markov failure/repair manufacturing systems. This paper has shown that under such policies, the stock level of manufacturing systems that produce a single part-type can be obtained even when failure rates of the machine depend on the production rate of parts. From the numerical study it has been found that for two parallel machines systems, when the failure rate of the main machine depends on its production rate, the hedging point policies are optimal among feedback policies and the reliability of the machines is enhanced. This result generalizes the results of Akella and Kumar (1986) which are derived for a constant failure rate and the works of Martinelli (2010) which is derived for a single machine with production rate dependent failure rate. To optimise the production policies, an experimental approach based on design of experiments, simulation modelling and response surface methodology has been used. The usefulness of the proposed approach is outlined for more complex situations in which analytical solutions are not easy to obtain. In the future, we plan to extend the proposed model to the reverse logistics (a hybrid manufacturing and remanufacturing system) with production rate dependent failure rates of the remanufacturing machine.

APPENDIX A. OPTIMALITY CONDITIONS AND NUMERICAL APPROACH

The properties of the value function and the manner in which the Hamilton-Jacobi-Bellman (HJB) equations are obtained can be found in Martinelli (2010). He describes the optimal control policies (optimality conditions) for one-machine manufacturing system with production rate dependent failure rates. Regarding the optimality principle, we can write the HJB equations as follows:

$$\rho v(\alpha,x) = \min_{(u_1,u_2)\in A(\alpha)} \left[\begin{array}{l} g(\alpha,x,u_1,u_2) + \sum_{\beta\in B} q^{\theta}_{\alpha\beta} v(\beta,x) + \\ (u_1+u_2-d)\dfrac{\partial v(\alpha,x)}{\partial x} \end{array} \right] \tag{A.1}$$

where $\dfrac{\partial v(\alpha,x)}{\partial x}$ is the partial derivatives of the value function $v(\alpha,x)$

The optimal control policies over $A(\alpha)$ of the right hand side of equation (A.1) are $(u_1^*(\cdot),u_2^*(\cdot))$. When the value function described by equation (9) is available, optimal control policies can be obtained as in equation (A.1).

To solve the HJB equations, the numerical method based on the Kushner (1992) approach as in Gharbi et al. (2011) and references therein is used. By approximating $v(\alpha,x)$ by a function $v^h(\alpha,x)$ and the first-order partial derivative of the value function $\dfrac{\partial v(\alpha,x)}{\partial x}$ by:

$$\frac{\partial v(x,\alpha)}{\partial x} = \begin{cases} \dfrac{1}{h}\left(v^h(\alpha,x+h)-v^h(\alpha,x)\right) & \text{if } (u_1+u_2-d)>0 \\ \dfrac{1}{h}\left(v^h(\alpha,x)-v^h(\alpha,x-h)\right) & \text{otherwise} \end{cases}$$

the HJB equation becomes:

$$v^h(x,\alpha) = \min_{(u_1,u_2)\in A(\alpha)} \left[\frac{ g(x,\alpha,u_1,u_2) + \sum_{\beta\neq\alpha} q^{\theta}_{\alpha\beta} v^h(x,\beta) + \dfrac{(u_1+u_2-d)}{h}\left[\begin{array}{l} v^h(x+h,\alpha)Ind\{u_1+u_2-d\geq 0\} \\ +v^h(x-h,\alpha)Ind\{u_1+u_2-d<0\} \end{array}\right] }{ \left(\rho + \dfrac{|u_1+u_2-d|}{h} + |q^{\theta}_{\alpha\alpha}|\right) } \right] \tag{A.2}$$

with $q^{\theta}_{\alpha\alpha} = -\sum_{\beta\neq\alpha} q^{\theta}_{\alpha\beta}$, $A^h(\alpha)$ is the numerical control grid and $Ind\{\Phi\} = \begin{cases} 1 & \text{if } \Phi \text{ is true} \\ 0 & \text{otherwise} \end{cases}$

The system of equations (A.2) can be interpreted as the infinite horizon dynamic programming equation of a discrete-time, discrete-state decision process, as in Boukas and Haurie (1990). In this paper, we use the value iteration procedure to approximate the value function given by equation (A.2). Charlot et al. (2007) and references therein provide details on such methods.

The discrete dynamic programming equation (A.2) gives the following six equations:

- **state 1**

$$v^h(x,1) = \begin{cases} \min\limits_{u_2 \in [0,u_{2\max}]} \left[\dfrac{ g(x,\alpha,u_1,u_2) + \dfrac{(u_1+u_2-d)}{h} \left[\begin{array}{l} v^h(x+h,1) Ind\{u_1+u_2-d \ge 0\} + \\ v^h(x-h,1) Ind\{u_1+u_2-d < 0\} \end{array} \right] + q_{12}^2 v^h(x,2) + q_{12}^{11} v^h(x,3) }{ \left(\rho + \dfrac{|u_1+u_2-d|}{h} + q_{12}^2 + q_{12}^{11} \right) } \right] & \text{if } u_1 \in (U, u_{1\max}] \\ \min\limits_{u_2 \in [0,u_{2\max}]} \left[\dfrac{ g(x,\alpha,u_1,u_2) + \dfrac{(u_1+u_2-d)}{h} \left[\begin{array}{l} v^h(x+h,1) Ind\{u_1+u_2-d \ge 0\} + \\ v^h(x-h,1) Ind\{u_1+u_2-d < 0\} \end{array} \right] + q_{12}^2 v^h(x,2) + q_{12}^{12} v^h(x,3) }{ \left(\rho + \dfrac{|u_1+u_2-d|}{h} + q_{12}^2 + q_{12}^{12} \right) } \right] & \text{if } u_1 \in [0, U] \end{cases}$$

(A.3)

- **state 2**

$$v^h(x,2) = \begin{cases} \min \left[\dfrac{ g(x,\alpha,u_1) + \dfrac{(u_1-d)}{h} \left[v^h(x+h,2) Ind\{u_1-d \ge 0\} + v^h(x-h,2) Ind\{u_1-d < 0\} \right] + q_{21}^1 v^h(x,1) + q_{12}^{11} v^h(x,4) }{ \left(\rho + \dfrac{|u_1-d|}{h} + q_{21}^2 + q_{12}^{11} \right) } \right] & \text{if } u_1 \in (U, u_{1\max}] \\ \min \left[\dfrac{ g(x,\alpha,u_1) + \dfrac{(u_1-d)}{h} \left[v^h(x+h,2) Ind\{u_1-d \ge 0\} + v^h(x-h,2) Ind\{u_1-d < 0\} \right] + q_{21}^2 v^h(x,1) + q_{12}^{12} v^h(x,4) }{ \left(\rho + \dfrac{|u_1-d|}{h} + q_{21}^2 + q_{12}^{12} \right) } \right] & \text{if } u_1 \in [0, U] \end{cases}$$

(A.4)

- state 3

$$v^h(x,3) = \min_{u_2 \in [0,u_{2\max}]} \left[\frac{ g(x,\alpha,u_2) + \dfrac{(u_2-d_2)}{h} \left[v^h(x+h,3) Ind\{u_2-d_2 \ge 0\} + v^h(x-h,3) Ind\{u_2-d_2 < 0\} \right] + q_{21}^1 v^h(x,1) + q_{12}^2 v^h(x,4) }{ \left(\rho + \dfrac{|u_2-d_2|}{h} + q_{21}^1 + q_{12}^2 \right) } \right] \quad (A.5)$$

- state 4

$$v^h(x,4) = \min\left[\frac{g(x,\alpha)+q_{21}^1 v^h(x,2)+q_{21}^2 v^h(x,3)+\frac{d}{h}v^h(x-h,4)}{\left(\rho+\frac{d}{h}+q_{21}^1+q_{21}^2\right)}\right] \tag{A.6}$$

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