H_∞ OPTIMAL CONTROL OF DISCRETE-TIME SINGULARLY PERTURBED SYSTEMS

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ABSTRACT

The proposed paper provides new algorithms for solving the H_∞-optimal control of discrete-time singularly perturbed systems by using the exact decomposition scheme. In one hand, we use the bilinear transformation to generate continous-time generalized algebraic Riccati equation. On the other hand, we take advantage of the results proposed in (Hsieh and Gajic 1998), and (Bidani, Radhy and Bensassi 2002), to derive new schemes for transforming the pure-slow and pure-fast nonsymmetric discrete-time Riccati equations (NDRE) into continuous-time ones. Two scheme are based on reducing the backward into forward Hamiltonian matrix (Bidani, Radhy and Bensassi 2002). The others ones use the bilinear transformation (Lim, Gajic and Shen 1995).

Keywords: Riccati equation, H_∞ optimal control, perturbation singular.

1. INTRODUCTION

The H_∞-control problems for singularly perturbed systems have been studied in different set-up from different point of view and have witnessed a fast growth development during this twentieth century. (Pan and Bassar 1993,1994) have studied the H_∞-control problem for singularly perturbed systems via a differential game-theoretic approach. (Dragon 1993, 1996) found the boundary of the H_∞-norm for singularly perturbed systems. All these works and others consider the $O(\epsilon)$ -approximation of the two-time scale discrete generalized Riccati equation. However, it is shown through (Pan and Bassar 1993,1994), if ϵ is sufficiently small, that the attenuation disturbance coefficient of the global system $\mu(\epsilon)$ converges to $Max(\mu_s, \mu_f)$,where μ_s and μ_f design respectively the "attenuation disturbance coefficient" of the slow and fast subsystems. Thus, if ϵ is not small enough the $O(\epsilon)$ -theory, used so far, in the paper (Pan and Bassar 1993) might not produce satisfactory results.

In order to broaden the applicable systems, the development of the $O(\epsilon^k)$ -theory is a necessary requirement. Some schemes have been given; Fridman (1995, 1996) discussed the near-optimal problem of singularly perturbed systems by using a high-order accuracy controller via Sobolev's concept of decomposition (Sobolev 1984). Hsieh and Gajic (1998) proposed the exact decomposition of continuous two-time scale algebraic Riccati equation into pure-slow and pure-fast non-symmetric continuous ones.

The same problem discussed in the continuous-time systems is encountered in the discrete-time version. The main facing problem is the stiffness of the two-time scale generalized discrete Riccati equation.

In the first scheme, we are contented by interpolating the resulting two-time scale generalized discrete Riccati equation into its counterpart continuous-time version. Henceforth, our purpose in this work consists in applying the bilinear interpolation to transform the generalized algebraic discrete Riccati equation of H_∞-optimal control of discrete-time two-time scale system into the corresponding two-time scale continuous-time one. Then we use the exact decomposition of the generalized algebraic continous Riccati equation into pure-slow and pure-fast nonsymmetric continous generalized algebraic Riccati equations.

The second scheme, discussed in this paper, is based on the use of the forward Hamiltonian form. Then according to this scheme, we obtain two pure-slow and pure-fast nonsymmetric discrete generalized algebraic Riccati equations.

The third scheme, in this paper, consists in using the forward form Hamiltonian system. This way provides two nonsymmetric continous generalized algebraic Riccati equations.

In the end, we use the bilinear interpolation to transform two pure-slow and pure-fast nonsymmetric discrete generalized algebraic Riccati equations, obtained in the third scheme, to the continous-time counterpart. And this constitute the fouth scheme.

Notation :

I_n : denotes matrix identity with dimension n .

$(O)^T$: denotes the transpose of a matrix.

$\|z\|^2=\sum_{n=0}^{\infty}(z^T(n)z(n))$: denotes the l_2 -norm of sequence $\{z(n)\}$.

T_{zw} : denotes the transfer function from ω to z .

2. STATEMENT PROBLEM

Consider a system governed by the following state equation

$$x(n+1)=Ax(n)+Bu(n)+Gw(n) \quad , x(0)=0 \qquad (1)$$

$$z(n+1)=Cx(n)+Du(n) \qquad (2)$$

The matrices A, B, C and D are partitioned as :

$$A=\begin{bmatrix} I_{n_1}+\epsilon A_{11} & \epsilon A_{12} \\ A_{21} & A_{22}\end{bmatrix}, \quad B=\begin{bmatrix}\epsilon B_1 \\ B_2\end{bmatrix}, \quad G=\begin{bmatrix}\epsilon G_1 \\ G_2\end{bmatrix},$$

$C=[\epsilon C_1 \quad C_2]$ and $D=\begin{bmatrix}\epsilon D_1 \\ D_2\end{bmatrix}$

where $x(n)\in\mathbb{R}^n$ is the state vector, $z(n)\in\mathbb{R}^p$ is the controlled output, $u(n)\in\mathbb{R}^m$ is the control vector, and $w(n)\in\mathbb{R}^l$ is the disturbance. In the sequel we assume the following :

1. (A, C) is detectable
2. (A, B) is stabilizable
3. $[C \quad D]^T[C \quad D]=[CC^T \quad DD^T]$ and that DD^T is a positive definite matrix.

Since it is difficult to find a control strategy $u(n)$ in l^2 -norm which minimizes the H_∞-norm of T_{zw}, we are quite content to find $u(n)$ in l^2 -norm that leads to $\|T_{zw}\|_\infty<\mu$ for a given constant $\mu>\mu_0$, where μ_0 denotes the minimum in the H_∞-norm of T_{zw}.

To this end, we call for the link that exists between the H_∞-optimal control and the linear quadratic difference games theory.

We state the discrete game as the suitable scheme for finding the sequences $u^*(n)$ and $w^*(n)$ that bring J ,

$$J=\tfrac{1}{2}\sum_{n=0}^{\infty}(z^T(n)z(n)-\mu^2w^T(n)w(n)) \\ =\tfrac{1}{2}(\|z\|^2-\mu^2\|w\|^2) \qquad (3)$$

to a saddle-point equilibrium. $u^*(n)$ is the lower value that minimize J and $w^*(n)$ is the upper value that maximize J.

First let us introduce an Hamiltonian function as :

$$H=\tfrac{1}{2}\left(x^T(n)C^TCx(n)+u^T(n)D^TDu(n)-\mu^2w^T(n)w(n)\right. \\ \left.+p^T(n+1)(Ax(n)+Bu(n)+Gw(n))\right) \qquad (4)$$

It can be verified that $H(x, p, u, w)$, considered as a function of two players (u^*, w^*) determined by the optimality conditions $\left(\frac{\partial H(x,p,u,w)}{\partial u}\right)_{(u,w)=(u^*,w^*)}=0$, $\left(\frac{\partial H(x,p,u,w)}{\partial w}\right)_{(u,w)=(u^*,w^*)}=0$.

The unique solution is provided by $u^*(n)=-(D^TD)^{-1}B^Tp(n+1)$, $w^*(n)=\frac{1}{\mu^2}G^Tp(n+1)$.

Then by considering the two other conditions $\left(\frac{\partial H(x,p)}{\partial x}\right)=p(n)$, $\left(\frac{\partial H(x,p)}{\partial p}\right)=x(n+1)$, it results in the corresponding Hamiltonian matrix form

$$\begin{bmatrix}x(n+1)\\ p(n)\end{bmatrix}=\begin{bmatrix}A & -\left(B(D^TD)^{-1}B^T-\frac{1}{\mu^2}GG^T\right)\\ C^TC & A^T\end{bmatrix}\begin{bmatrix}x(n)\\ p(n+1)\end{bmatrix} \qquad (5)$$

Thereafter, linearizing the co-state $p(n)$ with respect to $x(n)$, that is $p(n)=Px(n)$, we arrive to the saddle point

$$\begin{bmatrix}u^*(n)\\ w^*(n)\end{bmatrix}=-\left(\begin{bmatrix}D^TD & 0\\ 0 & -\mu^2I_l\end{bmatrix}+[B \quad G]^TP[B \quad G]\right)^{-1}[B \quad G]^TPAx(n) \qquad (6)$$

where the matrix P is determined by resolving the following H_∞-algebraic Riccati equation

$$P=A^TPA-A^TP[B \quad G]\left(\begin{bmatrix}D^TD & 0\\ 0 & -\mu^2I_l\end{bmatrix}+[B \quad G]^TP[B \quad G]\right)^{-1}[B \quad G]^TPA+C \qquad (7)$$

or equivalently

$$P=A^TP\left(I_n+\left(B(D^TD)^{-1}B^T-\mu^2GG^T\right)P\right)^{-1}A+C^TC$$

Here we should emphasis the fact that $w^*(n)$ represents the worst possible disturbance and henceforth it serves only for the design purpose. Against it $u^*(n)$ represents the optimal feedback control strategy that should be applied in practice. Therefore we should proceed by isolate $u^*(n)$ from $w^*(n)$.

To this end we use the identity $(\alpha+\delta\beta)^{-1}=\alpha^{-1}\delta(I+\beta\alpha^{-1}\delta)^{-1}$ for $\delta=[B \quad G]^TP$ and $\beta=[B \quad G]$ and $\alpha=\begin{bmatrix}D^TD & 0\\ 0 & -\mu^{-2}I_l\end{bmatrix}$, we then obtain

$$u^*(n)=(D^TD)^{-1}B^TP\left(I_n+\left(B(D^TD)^{-1}B^T-\mu^2GG^T\right)P\right)^{-1}Ax(n) \qquad (8)$$

$$w^*(n)=\mu^2G^TP\left(I_n+\left(B(D^TD)^{-1}B^T-\mu^2GG^T\right)P\right)^{-1}Ax(n) \qquad (9)$$

Notice here that the computation of $u^*(n)$ is inadequate because of the existence of the matrix inverse $(D^T D)$, and this matrix $(D^T D)$ is proportional to the parameter ϵ (considered as the fast discretizing time). So to avoid this problem, we use a change of variables. Letting define the new matrix $M=P\left(I_n-\mu^{-2}GG^T P\right)^{-1}$ so that

$$P\left(I_n+\left(B\left(D^T D\right)^{-1}B^T-\mu^2 GG^T\right)P\right)^{-1}=M\left(I_n+B\left(D^T D\right)^{-1}B^T M\right)^{-1}$$

Then using the identity $(\alpha+\delta\beta)^{-1}=\alpha^{-1}\delta\left(I+\beta\alpha^{-1}\delta\right)^{-1}$,for $\alpha=D^T D$ and $\beta=B$, the expression in (8) is reduced into

$$u^*(n)=-\left(D^T D+B^T MB\right)^{-1}B^T MAx(n) \qquad (10)$$

Hence, once we apply the feedback control (10), the signal $w(n)$ becomes irrelevant for the stability analysis if $w(n)\leqslant w^*(n)$.

Notice that requirement $\left(\mu^2 I_l-G_T PG\right)>0$ should be added to the generalized algebraic Riccati equation (7); otherwise the optimal $w^*(n)$ or the worst-case disturbance becomes unbounded which implies in turn that the stabilizing solution P of the generalized algebraic Riccati equation (7) does not exist (see for more details (Basar 1991)).

To determine the closed-loop information pattern, we call for the following theorem :

Theorem 1: *Consider the system (1)-(2) and assume that the triplet $\left(A, B, \sqrt{C^T C}\right)$ is stabilizable-detectable. Then the following are equivalent :*

- *There exists a feedback law $u=Kx$ which stabilizes the system (1)-(2) and renders the H_∞-norm of the transfer matrix T_{zw} strictly less than $\|T_{zw}\|_\infty<\mu$.*
- *There exists a symmetric positive semi-definite stabilizing solution $P\geq 0$ satisfying the generalized algebraic Riccati equation (7) and the inequality $\left(\mu^2 I_l-G^T PG\right)>0$.*

Moreover, one such controller is $K=-\left(D^T D+B^T MB\right)^{-1}B^T MAx(n)$ □

Proof. see **Appendix A** □

3. NEW ALGORITHMS FOR SOLVING H_∞ OPTIMAL CONTROL OF DISCRETE-TIME SINGULARLY PERTURBED SYSTEMS

3.1. First scheme

Assumption 4 : *The matrix $\left(I_n+A\right)$ is invertible* □

Remarque : *The assumption 4 is not a limited problem since we can make an input $u=Fx+v$ such that $(A+F)$satisfies assumption 4*□

As we see the presence of a small positive parameter ϵ in matrices A, B makes the resolution of the generalized algebraic Riccati equation (7) ill-conditioned and with a way quite analogous to the partitioning in the standard regulator problem the matrix $P=P(\epsilon)$ is partitioned as : $\begin{bmatrix} P_1/\epsilon & P_2 \\ P_2^T & P_3 \end{bmatrix}$

Then, the substitution of the latter structure into the generalized algebraic Riccati equation (7) results in a more complicated equations and the reduced of the computation becomes limited. (Lim, Gajic and Shen 1995) had discussed the analogue regulator problem and they had proposed the use of a bilinear interpolation to transform the discrete-time Riccati equation into a continuous-time one. Thus, we should use the same technique to transform the generalized discrete-time Riccati equation (7) into a generalized continuous-time Riccati one.

Lemma 2 : *There exists a generalized continuous Riccati equation $PA_c+A_c^T P+Q_c-P\left(B_c R_c^{-1}B_c-Z_c\right)P=0$ corresponding to the generalized discrete Riccati equation* (7).

Under the bilinear transformation, matrices A_c, B_c, Q_c, R_c and Z_c are deduced as

$$A_c=\left(I_n-2\Phi^{-T}\right), Q_c=2\Phi\left(C^T C\right)\left(I_n+A\right)^{-1};$$
$$\Phi=\left(I_n+A^T\right)+\left(C^T C\right)\left(I_n+A\right)^{-1}\left(B\left(D^T D\right)^{-1}B^T-\mu^2 GG^T\right);$$
$$B_c=\left(I_n-\left(I_n+A\right)^{-1}\mu^{-2}GG^T\left(I_n+A\right)^{-T}C^{-T}C\right)^{-1}\left(I_n+A\right)^{-1}B;$$

$$R_c=\frac{1}{2}D^T D+\frac{1}{2}B^T\left(I_n+A\right)^{-T}\times$$
$$\left(I_n-\left(C^T C\right)\left(I_n+A\right)^{-1}\mu^{-2}GG^T\left(I_n+A\right)^{-T}\right)^{-1}\left(C^T C\right)\left(I_n+A\right)^{-1}B;$$
$$Z_c=\frac{2}{\mu^2}\left(I_n+A\right)^{-1}G\left(I_n-\frac{1}{\mu^2}G^T\left(I_n+A\right)^{-T}\left(C^T C\right)\left(I_n+A\right)^{-1}G\right)^{-1}\times$$
$$G^T\left(I_n+A\right)^{-T}$$

□

Proof. see **Appendix B**□

To accomplish this scheme, we introduce the following requirement

$$I_l-\frac{1}{\mu^2}G^T\left(I_n+A\right)^{-T}\left(C^T C\right)\left(I_n+A\right)^{-1}G>0 \quad (11)$$

in order that we keep matrices $B_c R_c^{-1}B_c^T$ and Z_c positive definite. At the first glance, one see that this condition proves the fact that the digital control is more robust than its corresponding analogy one.

Then if we consider that μ satisfies the latter requirement, the stabilizing solution P is derived from the following continuous generalized algebraic Riccati equation

$$P A_c + A_c^T P + Q_c - P\left(B_c R_c^{-1} B_c^T - Z_c\right)P = 0 \qquad (12)$$

With $\mu^2 I_l - G^T P G > 0$ and

$I_l - \frac{1}{\mu^2} G^T \left(I_n + A\right)^{-T} \left(C^T C\right)\left(I_n + A\right)^{-1} G > 0.$

But the computation of this equation is also stiff since A_c has the form $\begin{bmatrix} \epsilon A_{c11} & \epsilon A_{c12} \\ A_{c21} & A_{c22} \end{bmatrix}$ and B_c the form $\begin{bmatrix} \epsilon B_{c1} \\ B_{c2} \end{bmatrix}$ and Z_c the form $\begin{bmatrix} \epsilon^2 Z_{c11} & \epsilon Z_{c12} \\ \epsilon Z_{c12}^T & Z_{c22} \end{bmatrix}$.

To resolve this equation, we refer to the paper (Hsieh and Gajic 1998). Notice that if ϵ is not sufficiently small, we use the Schur vector method, instead of Newton iterative method, to resolve the resulting non-symmetric continuous generalized algebraic Riccati equations.

3.2. Second scheme

Using the same technique than in (Bidani, Radhy and Bensassi 2002) and by imposing $p(n) = \begin{bmatrix} p_1/\epsilon \\ p_2 \end{bmatrix}$, with $p_1 \in \mathbb{R}^{n_1}$ and $p_2 \in \mathbb{R}^{n_2}$, and interchanging the second and the third rows in (8), we obtain

$$\begin{bmatrix} x_1(n+1) \\ p_1(n) \\ x_2(n+1) \\ p_2(n) \end{bmatrix} = \begin{bmatrix} I_{2n_1} + \epsilon T_1 & \epsilon T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} x_1(n) \\ p_1(n+1) \\ x_2(n) \\ p_2(n+1) \end{bmatrix}$$

where $T_1 = \begin{bmatrix} A_{11} & -(B_1 R^{-1} B_1^T - \frac{1}{\mu^2} G_1 G_1^T) \\ Q_1 & A_{11}^T \end{bmatrix}$,

$T_2 = \begin{bmatrix} A_{12} & -(B_1 R^{-1} B_2^T - \frac{1}{\mu^2} G_1 G_2^T) \\ Q_2 & A_{21}^T \end{bmatrix}$,

$T_3 = \begin{bmatrix} A_{21} & -(B_2 R^{-1} B_1^T - \frac{1}{\mu^2} G_2 G_1^T) \\ Q_2^T & A_{12}^T \end{bmatrix}$,

$T_4 = \begin{bmatrix} A_{22} & -(B_2 R^{-1} B_2^T - \frac{1}{\mu^2} G_2 G_2^T) \\ Q_3 & A_{22}^T \end{bmatrix}$ and

$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}$

On the other side, the change of the original states $\begin{bmatrix} x_1(n) \\ p_1(n) \\ x_2(n) \\ p_2(n) \end{bmatrix}$ to the new ones $\begin{bmatrix} \eta_1(n) \\ \xi_1(n) \\ \eta_2(n) \\ \xi_2(n) \end{bmatrix}$, with the help of the Chang matrix defined by

$$\begin{bmatrix} \eta_1(n) \\ \xi_1(n+1) \\ \eta_2(n) \\ \xi_2(n+1) \end{bmatrix} = K \begin{bmatrix} x_1(n) \\ p_1(n+1) \\ x_2(n) \\ p_2(n+1) \end{bmatrix} \quad \text{with}$$

$K = \begin{bmatrix} I_{2n_1} - \epsilon H L & -\epsilon H \\ L & I_{2n_1} \end{bmatrix}$, $K^{-1} = \begin{bmatrix} I_{2n_1} & -\epsilon H \\ -L & I_{2n_1} - \epsilon L H \end{bmatrix}$,

derives the pure slow and pure fast sub-Hamiltonians respectively

$$\begin{bmatrix} \eta_1(n+1) \\ \eta_2(n) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} \eta_1(n) \\ \eta_2(n+1) \end{bmatrix} \qquad (13)$$

$$\begin{bmatrix} \xi_1(n+1) \\ \xi_2(n) \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} \xi_1(n) \\ \xi_2(n+1) \end{bmatrix} \qquad (14)$$

where $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = I_{2n_1} + \epsilon(T_1 - T_2 L)$ and $\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = T_4 + \epsilon L T_2$ if the following Chang equations are satisfied

$$(I_{2n_2} - T_4) L + T_3 + \epsilon L (T_1 - T_2 L) = 0 \qquad (15)$$

$$H(I_{2n_2} - T_4 - \epsilon L T_2) + T_2 + \epsilon (T_1 - T_2 L) H = 0 \qquad (16)$$

Assumption 5 : $(I_{n_2} - A_{22})$ is non-singular through-out this paper.

Under the assumption5 or more precisely the assumption that the matrix $(I_{2n_2} - T_4)$ is non-singular, different known techniques are used to solve equations (15)(16). Here are the frame of references : the fixed point method , Newton method, the asymptotic expansion and Taylor series methods, and finally the eigenvector approach.

Therefore, the linearization of the new co-states η_2 and ξ_2 with respect to η_1 and ξ_1, respectively,

$$\begin{bmatrix}\eta_2(n+1)\\ \xi_2(n+1)\end{bmatrix}=\begin{bmatrix}P_{rs}(I_{n_1}-a_2P_{rs})^{-1}a_1 & 0\\ 0 & P_{rf}(I_{n_2}-b_2P_{rf})^{-1}b_1\end{bmatrix}\begin{bmatrix}\eta_1(n)\\ \xi_1(n)\end{bmatrix} \quad (17)$$

reduced the pure slow and pure fast sub-Hamiltonians, previously defined, to the closed-loop form of two, completely decoupled, pure slow and pure fast subsystems respectively,

$$\eta_1(n+1)=(a_1+a_2P_{rs}(I_{n_1}-a_2P_{rs})^{-1}a_1)\eta_1(n) \quad (18)$$

$$\xi_1(n+1)=(b_1+b_2P_{rf}(I_{n_2}-b_2P_{rf})^{-1}b_1)\xi_1(n) \quad (19)$$

together with two reduced-order nonsymmetric algebraic discrete-time Riccati equations :

$$P_{rs}=a_4P_{rs}a_1+a_3+a_4P_{rs}(I_{n_1}-a_2P_{rs})^{-1}a_2P_{rs}a_1 \quad (20)$$

$$P_{rf}=b_4P_{rf}b_1+b_3+b_4P_{rf}(I_{n_{12}}-b_2P_{rf})^{-1}b_2P_{rf}b \quad (21)$$

The solution P_{rs} (respt. P_{rf}) of the equation (20) (resp. (21)) is deduced from following lemmas.

Assumption 6: the fast subsystem $(A_{22}, B_2, \sqrt{Q_3})$ is stabilizable-detectable.

Let $\mu_f = inf\{\mu>0\}/$ the fast discrete-time Riccati equation (21) has a positive definite solution.

Lemma 7 : Under the assumption 6 there exists $\epsilon_1>0$ such that for any $\epsilon>\epsilon_1$ an unique solution of (21) exists.

Proof.

By using the first approximation in ϵ of b_i $(i=1,2,3,4)$, it results in

$$\begin{bmatrix}b_1 & b_2\\ b_3 & b_4\end{bmatrix}=\begin{bmatrix}A_{22} & -(B_2R^{-1}B_2^T-\frac{1}{\mu^2}G_2G_2^T)\\ Q_3 & A_{22}^{22}\end{bmatrix}$$ yielding in turn the symmetric discrete-time Riccati equation :

$$P_{rf}=Q_3+A_{22}^TP_{rf}(I_{n_2}+(B_2R^{-1}B_2^T-\frac{1}{\mu^2}G_2G_2^T)P_{rf})^{-1}A_{:} \quad (22)$$

Therefore the use of Lemma 3 dictates that the unique solution P_{rf} of the equation (22) exists if the system $(A_{22}, B_2, \sqrt{(Q_3)})$ is stabilizable-detectable and the corresponding transfer matrix is inferior to certain μ_f.

To accomplish this proof, the implicit function theorem (Bidani, Radhy and Bensassi 2002) guaranteed the existence and uniqueness of the solution of equation (22) for $\epsilon\leqslant\epsilon_1$ ■

Assumption 8: The slow subsystem $(A_o, \sqrt{(B_oR_o^{-1}B_o^T)}, C_o)$ is stabilizable-detectable with

$$A_o=I_{n_1}+\epsilon(A_{11}+A_{12}(I_{n_2}-A_{22})^{-1}A_{21}),\ R_o=R+D_o^TD_o$$

$$C_o=C_1+C_2(I_{n_2}-A_{22})^{-1}A_{21},\qquad D_o=C_2(I_{n_2}-A_{22})^{-1}B_2$$

and $B_o=\epsilon(B_1+A_{12}(I_{n2}-A_{22})^{-1}B_2)$ ■

Notice that **assumption 8** uses the fact that C_1 is full-rank factorization of Q_1 (i.e. $Q_1=C_1^TC_1$) and C_1 is full-rank factorization of Q_3 (i.e. $Q_3=C_2^TC_2$).

Since $A_o-\sqrt{(B_oR_o^{-1}B_o^T)}K$ is stable by hypothesis, the pair $(A_o, \frac{1}{\mu}G_o)$ is, indeed, stabilizable for $\mu>\mu_s$ where $G_o=\epsilon(G_1+A_{22}(I_{n_2}-A_{22})^{-1}G_2)$.

Lemma 9 : Under the assumption 8 there exists $\epsilon>0$ such that for any $\epsilon\leqslant\epsilon_2$ an unique solution of (20) exists.

Proof :

The proof is the same than these used in the paper (Bidani, Radhy and Bensassi 2002) the only change is that of Lemma 3 to state that the unique solution P_{rs} of the equation

$$P_{rs}=C_o^TC_o+A_o^TP_{rs}(I_{n_1}+(B_oR_o^{-1}B_o^T-\frac{1}{\mu^2}G_oG_o^T)P_{rs})^{-1}A_o$$

exists if the system $(A_o, \sqrt{(B_oR_o^{-1}B_o^T)}, C_o)$ is stabilizable-detectable and the corresponding transfer matrix is inferior to a certain μ_s. Then the use of the implicit function theorem (Bidani, Radhy and Bensassi 2002) guaranteed the existence and uniqueness of the solution of equation (20) for $\epsilon\leqslant\epsilon_2$ ■

3.3. Third scheme

Assumption 10 : The matrix A_{22} is non-singular.

Under Assumption 10, one can transform the pure-slow and pure-fast backward sub-Hamiltonians form (13)(14) into the equivalent pure slow and pure fast forward sub-Hamiltonians form, respectively, see (Bidani, Radhy and Bensassi 2002), (Lim, Gajic and Shen 1995) and (Hsieh and Gjaic 1998). Therefore the transformation of nonsymmetric algebraic discrete-time Riccati equations (20)(21) into nonsymmetric continuous-time algebraic Riccati equations are deduced straight-away,

$$P_{rs}\bar{a}_1-\bar{a}_4 P_{rs}-\bar{a}_3+P_{rs}\bar{a}_2 P_{rs}=0$$
$$P_{rf}\bar{b}_1-\bar{b}_4 P_{rf}-\bar{b}_3+P_{rf}\bar{b}_2 P_{rf}=0$$

with $\begin{bmatrix}\bar{a}_1 & \bar{a}_2\\ \bar{a}_3 & \bar{a}_4\end{bmatrix}=\begin{bmatrix}(a_1-a_2 a_4^{-1} a_3) & a_2 a_4^{-1}\\ -a_4^{-1} a_3 & a_4^{-1}\end{bmatrix}$,

$$\begin{bmatrix}\bar{b}_1 & \bar{b}_2\\ \bar{b}_3 & \bar{b}_4\end{bmatrix}=\begin{bmatrix}(b_1-b_2 b_4^{-1} b_3) & b_2 b_4^{-1}\\ -b_4^{-1} b_3 & b_4^{-1}\end{bmatrix}$$

Using permutation matrices E_1, E_2, E_3 and E_4:

$$E_1=\begin{bmatrix}I_{n_1} & 0 & 0 & 0\\ 0 & 0 & \epsilon I_{n_1} & 0\\ 0 & I_{n_2} & 0 & 0\\ 0 & 0 & 0 & I_{n_2}\end{bmatrix},\quad E_2=\begin{bmatrix}I_{n_1} & 0 & 0 & 0\\ 0 & 0 & I_{n_2} & 0\\ 0 & I_{n_1} & 0 & 0\\ 0 & 0 & 0 & I_{n_2}\end{bmatrix}$$

$$E_1=\begin{bmatrix}I_{n_1} & 0 & 0 & 0\\ a_3 & a_4 & 0 & 0\\ 0 & I_{n_2} & 0 & 0\\ 0 & 0 & b_3 & b_4\end{bmatrix},$$

$$E_1=\begin{bmatrix}I_{n_1} & 0 & 0 & 0\\ \epsilon Q_1 & (I_{n_1}+\epsilon A_{11}^T) & \epsilon Q_2 & \epsilon A_{21}^T\\ 0 & 0 & I_{n_2} & 0\\ Q_2^T & A_{12}^T & Q_3 & A_{22}^T\end{bmatrix}$$

defined as follow

$$\begin{bmatrix}x_1(n)\\ p_1(n)\\ x_2(n)\\ p_2(n)\end{bmatrix}=E_1\begin{bmatrix}x_1(n)\\ x_2(n)\\ p_1(n)\\ p_2(n)\end{bmatrix},\quad \begin{bmatrix}\eta_1(n)\\ \xi_1(n)\\ \eta_2(n)\\ \xi_2(n)\end{bmatrix}=E_2\begin{bmatrix}\eta_1(n)\\ \eta_2(n)\\ \xi_1(n)\\ \xi_2(n)\end{bmatrix}$$

$$\begin{bmatrix}\eta_1(n)\\ \eta_2(n)\\ \xi_1(n)\\ \xi_2(n)\end{bmatrix}=E_3\begin{bmatrix}\eta_1(n)\\ \eta_2(n+1)\\ \xi_1(n)\\ \xi_2(n+1)\end{bmatrix},\quad \begin{bmatrix}x_1(n)\\ p_1(n)\\ x_2(n)\\ p_2(n)\end{bmatrix}=E_4\begin{bmatrix}x_1(n)\\ p_1(n+1)\\ x_2(n)\\ p_2(n+1)\end{bmatrix}$$

Taking into account of the results used in (Bidani, Radhy and Bensassi 2002), we derive the transformation matrices $\Pi=E_2 E_3 K E_4^{-1} E_1$, $\Phi=E^{-1} E_4 K^{-1} E_3^{-1} E_2$ leading thereafter to

$$\begin{bmatrix}\eta_1(n)\\ \xi_1(n)\end{bmatrix}=(\Pi_1+\Pi_2 P)\,x(n),$$

$$\begin{bmatrix}\eta_2(n)\\ \xi_2(n)\end{bmatrix}=(\Pi_3+\Pi_4 P)\,x(n)$$, and to

$$x(n)=\left(\Phi_1+\Phi_2\begin{bmatrix}P_{rs} & 0\\ 0 & P_{rf}\end{bmatrix}\right)\begin{bmatrix}\eta_1(n)\\ \xi_1(n)\end{bmatrix}$$ and

$$P=\left(\Phi_3+\Phi_4\begin{bmatrix}P_{rs} & 0\\ 0 & P_{rf}\end{bmatrix}\right)\left(\Phi_1+\Phi_2\begin{bmatrix}P_{rs} & 0\\ 0 & P_{rf}\end{bmatrix}\right)^{-1}$$

with $\Pi=\begin{bmatrix}\Pi_1 & \Pi_2\\ \Pi_3 & \Pi_4\end{bmatrix}$, $\Phi=\begin{bmatrix}\Phi_1 & \Phi_2\\ \Phi_3 & \Phi_4\end{bmatrix}$.

4. CONCLUSION

In This paper, we have presented third scheme and fourth scheme is deduced by applying bilinear interpolation. The main facing problem to tackle is the resolution of the pure-slow and pure-fast nonsymmetric continuous generalized algebraic Riccati equations. To resolve this kind of equations, we use following the smallness of the perturbation parameter, ϵ, the iterative methods, for instance Newton method, or eigenvector and schur approach methods. As known the iterative methods are preferred for large scale systems. So, we conclude that the second scheme is not fast as the other schemes but it requires less memory.

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APPENDIX A.

In this appendix we call for an important property of linear systems, that relates the estimation of the so-called the H_∞-norm of the transform matrix of a system to the existence of solutions of an appropriate Riccati equation under stabilizability-dectectability assumption.

To this end, we consider a system described by the equation of the form

$$x(n+1)=A\,x(n)+B\,u(n) \qquad \text{(A.1)}$$

$$z(n)=C\,x(n) \qquad \text{(A.2)}$$

under the assumption that the system (A,B,C) is stabilizable-detectable, and without loss of generality, there exists a matrix T that transforms the system matrices (A,B,C) to the form

$\tilde{A}=T^{-1}A\,T=\begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ 0 & \tilde{A}_c\end{bmatrix}$, $\tilde{B}=T^{-1}B=\begin{bmatrix}\tilde{B}_1\\ 0\end{bmatrix}$ and $\tilde{C}=C\,T=\begin{bmatrix}\tilde{C}_1 & 0\end{bmatrix}$ where the subsystem $(\tilde{A}_{11},\tilde{B}_1,\tilde{C}_1)$ is controllable-observable and the matrix $\tilde{A}_c$ is asymptotically stable.

The matrices $A,\tilde{A}_{11},\tilde{B}_1,\tilde{C}_1,B$ and C are of dimension $n\times n,\bar{n}\times\bar{n},\bar{m}\times\bar{n},\bar{p}\times\bar{n},p\times n$ and $m\times n$ respectively.

Corollary 3 (A.1) (Goodwin and Sin 1984; Yaesh, and Shaked 1991): *Given that* $T_{zw}(z)=\tilde{C}_1\left(zI_{\bar{n}}-\tilde{A}_{11}\right)^{-1}\tilde{B}_1$ *of (A.1)-(A.2) is asymptotically stable then* $\|T_{zw}\|_\infty<\mu$ *if and only if there exists a positive definite solution to the following two equations*

$$\bar{P}_1=\tilde{A}_{11}^T\bar{P}_1\tilde{A}_{11}+\tilde{C}_1^T\tilde{C}_1+\tilde{A}_{11}^T\bar{P}_1\tilde{B}_1\left(\mu^2I_m-\tilde{B}_1^T\bar{P}_1\tilde{B}_1\right)^{-1}\tilde{B}_1^T\bar{P}_1\tilde{A}_{11}$$

and $\mu^2I_m-\tilde{B}_1^T\bar{P}_1\tilde{B}_1>0$ ■

Since

$$T_{zw}(z)=\tilde{C}_1\left(zI_{\bar{n}}-\tilde{A}_{11}\right)^{-1}\tilde{B}_1=\tilde{C}\left(zI_{\bar{n}}-\tilde{A}\right)^{-1}\tilde{B}=C\left(z\,I_n-A\right)^{-1}B$$

and the fact that $P=T^{-T}\bar{P}T^{-1}$ and $\bar{P}=\begin{bmatrix}\bar{P}_1 & \bar{P}_2\\ \bar{P}_2^T & \bar{P}_3\end{bmatrix}$ we deduce from the computation that $\bar{P}=\begin{bmatrix}\bar{P}_1 & 0\\ 0 & \bar{0}\end{bmatrix}$ is the unique solution.

Then the solution of the original equation $P=A^TPA+A^TPB\left(\mu^2I_m-B^TPB\right)^{-1}B^TPA+C^TC$ and $\mu^2I_m-B^TP\,B>0$ is straight-away obtained by which implies in turn that P is a symmetric positive semi-definite solution. Therefore we can state the new version of the latter corollary.

Corollary 4 ***(A.2)***: *Consider* $T_{zw}(z)=C(zI_n-A)^{-1}B$ *of (A.1)-(A.2) then* $\|T_{zw}(z)\|_\infty<\mu$ *if and only if there exists a symmetric positive semi-definite solution to the following two equations*

$$P=A^T PA+A^T PB(\mu^2 I_m-B^T PB)^{-1}B^T PA+C^T C \quad \text{(A.3)}$$

$$\mu^2 I_m-B^T PB>0 \quad \text{(A.4)}$$

□

Thereafter, we are now ready to proof the *Theorem 1.*

Proof. Suppose there exists a feedback law of the form $u=Kx$ that stabilizes the closed loop system :

$$x(n+1)=(A+BK)x(n)+Gw(n) \ \ , x(0)$$
$$z(n)=(C+DK)x(n)$$

and renders its l_2-gain strictly less than μ or equivalently $\|(C+DK)(zI_n-(A+BK)G)\|_\infty<\mu$. Then, by Corollary A2, similar equations (A.3)-(A.4) are satisfied by some symmetric matrix $P\geqslant 0$ satisfying

$$P=(A+BK)^T P(A+BK)+(A+BK)^T PG(\mu^2 I_l-G^T PG)^{-1}\times G^T P(A+BK)+(C+DK)^T(C+DK)$$ or

equivalently

$$P=(A+BK)^T P(\mu^2 I_l-GG^T P)^{-1}P(A+BK)+(C+DK)^T(C+DK)$$

and

$$(\mu^2 I_l-B^T PB)>0$$

On the other hand the representation of z in $l_2[0 \quad \infty]$ conducts to $(C+DK)^T(C+DK)\equiv(C^T C+K^T(D^T D)K)$ (see assumption 3).

Denoting by $M=P(I_n-\mu^{-2}GG^T P)^{-1}$, it follows that

$$(I_n+\mu^{-2}GG^T P)^{-1}M=(C^T C)+K^T(D^T D)K+(A+BK)^T M(A+BK)$$

Then we use the identities

$$(I_n+\mu^{-2}MGG^T)^{-1}=I_n-(I_n+\mu^{-2}MGG^T)^{-1}\mu^{-2}MGG^T$$

and $(I_n+\mu^{-2}MGG^T)^{-1}\mu^{-2}MGG^T M=\mu^{-2}MG(I_l+\mu^{-2}G^T MG)^{-1}G^T M$

to obtain $M=(A+BK)^T M(A+BK)$
$$+MG(\mu^2 I_l+G^T MG)^{-1}G^T M+(C)^T(C)+K^T(D)^T(D)K$$

and then $M=A^T MA+MG(\mu^2 I_l+G^T MG)^{-1}G^T M$
$$-A^T MB(D^T D+B^T MB)^{-1}B^T MA+C^T C+S$$ where

$$S=(K^T+A^T MB(D^T D+B^T MB)^{-1})(D^T D+B^T MB)\times(K+(D^T D+B^T MB)^{-1}B^T MA)$$

Suppose then that $u\neq u^*(S\neq 0)$, with $S>0$, we conclude according to the classical two-person zero-sum dynamic game that $(u=Kx, w^*)$ constitutes an other saddle-point and hence we can obtain for different value of K an infinity of saddle points. Contradiction (u^*, w^*) is the unique saddle point, hence $K=-(D^T D+B^T MB)^{-1}B^T MA$.

Now suppose that (7) is satisfied together with $\mu^2 I_l-G^T PG>0$ by $P\geqslant 0$ and choose $K=-(D^T D+B^T MB)^{-1}B^T MA$. The generalized algebraic Riccati equation is reduced into the form $P=(A+BK)^T P(\mu^2 I_m-GG^T P)^{-1}(A+BK)+C^T C+K^T D^T DK$

Then by the use of **corollary A.**2, we obtain $\|T_{zw}\|_\infty<\mu$ if we prove that the pair $(A+BK, \sqrt{C^T C+K^T D^T DK})\equiv(A+BK, C+DK)$ is detectable.

Hence suppose that v is an unstable eigenvector of $(A+BK)$ that corresponds to the unstable eigenvalue λ, and that it belongs to the kernel of $(C+DK)$, that is

$(A+BK)v=\lambda v$ and $(C+DK)v=0$.

Using the assumption that the triple (A, B, C) is stabilizable-detectable we conclude that $v=0$. Therefore the pair $\left(A+BK, \sqrt{C^T C+K^T D^T DK}\right)$ is detectable. This completes the proof of Theorem ■

APPENDIX B.

Proof. : Considers the following transformations :

$A_c=I_n-2\boldsymbol{F}^{-\boldsymbol{T}}, Q_c=2\boldsymbol{F}^{-1}C^T C\left(I_n+A\right)^{-1}$,

$Z=GG^T\ S=B\left(D^T D\right)^{-1}B^T$,

$\boldsymbol{F}=\left(I_n+A^T\right)+\left(C^T C\right)\left(I_n+A\right)^{-1}\left(S-\mu^{-2}Z\right)\left(S-\mu^{-2}Z\right)\boldsymbol{F}^{-1}\left(I_n+A^T\right)$

$S_c=2\left(I_n+A\right)^{-1}\left(S-\mu^{-2}Z\right)\boldsymbol{F}^{-1}$ and

$=\left(S-\mu^{-2}Z\right)\left(\left(I_n+A^T\right)^{-1}\boldsymbol{F}\right)^{-1}$.

Using the identity $(I+\delta\beta)^{-1}\delta=\delta(I+\beta\delta)^{-1}$ and the fact that

$(S-\mu^{-2}Z)=\begin{bmatrix}B & G\end{bmatrix}\begin{bmatrix}D^T D & 0\\ 0 & -\mu^2 I_l\end{bmatrix}^{-1}\begin{bmatrix}B^T\\ G^T\end{bmatrix}, \beta=\begin{bmatrix}B & G\end{bmatrix}$

and $\delta=\begin{bmatrix}B^T\\ G^T\end{bmatrix}$.

We obtain

$$S_c=2\begin{bmatrix}B_c & G_c\end{bmatrix}\begin{bmatrix}D^T D & 0\\ 0 & -\mu^2 I_l\end{bmatrix}^{-1}\times$$

$$\left(I_{l+m}+\begin{bmatrix}B^T\\ G^T\end{bmatrix}C^T C\begin{bmatrix}B_c & G_c\end{bmatrix}\begin{bmatrix}D^T D & 0\\ 0 & -\mu^2 I_l\end{bmatrix}^{-1}\right)^{-1}\begin{bmatrix}B^T\\ G^T\end{bmatrix}$$

with $\begin{bmatrix}B^T\\ G^T\end{bmatrix}=\left(I_n+A\right)^{-1}\begin{bmatrix}B & G\end{bmatrix}$.

The identity $(I+\beta)^{-1}=I-(I+\beta)^{-1}\beta$ for

$\beta=\begin{bmatrix}B^T\\ G^T\end{bmatrix}C^T C\begin{bmatrix}B_c & G_c\end{bmatrix}\begin{bmatrix}D^T D & 0\\ 0 & -\mu^2 I_l\end{bmatrix}^{-1}$ yields the

equality $S_c=2\begin{bmatrix}B_c & G_c\end{bmatrix}\begin{bmatrix}D^T D & 0\\ 0 & -\mu^2 I_l\end{bmatrix}^{-1}\begin{bmatrix}B^T\\ G^T\end{bmatrix}$

$$-2\begin{bmatrix}B_c & G_c\end{bmatrix}\begin{bmatrix}D^T D & 0\\ 0 & -\mu^2 I_l\end{bmatrix}^{-1}\times$$

$$\left(I_{l+m}+\begin{bmatrix}B^T\\ G^T\end{bmatrix}C^T C\begin{bmatrix}B_c & G_c\end{bmatrix}\begin{bmatrix}D^T D & 0\\ 0 & -\mu^2 I_l\end{bmatrix}^{-1}\right)^{-1}\times$$

$$\begin{bmatrix}B^T\\ G^T\end{bmatrix}C^T C\begin{bmatrix}B_c & G_c\end{bmatrix}\begin{bmatrix}D^T D & 0\\ 0 & -\mu^2 I_l\end{bmatrix}^{-1}.$$

Hence using the identity :

$(I+\delta\beta)^{-1}\delta=\delta(I+\beta\delta)^{-1}$ for $\delta=\begin{bmatrix}B^T\\ G^T\end{bmatrix}C^T$ and

$\beta=C\begin{bmatrix}B_c & G_c\end{bmatrix}\begin{bmatrix}D^T D & 0\\ 0 & -\mu^2 I_l\end{bmatrix}^{-1}$ we obtain

$$S_c=2\begin{bmatrix}B_c & G_c\end{bmatrix}\begin{bmatrix}D^T D & 0\\ 0 & -\mu^2 I_l\end{bmatrix}^{-1}\begin{bmatrix}B^T\\ G^T\end{bmatrix}$$
$$-2\begin{bmatrix}B_c & G_c\end{bmatrix}\begin{bmatrix}D^T D & 0\\ 0 & -\mu^2 I_l\end{bmatrix}^{-1}\times$$

$$\left(I_p+C\begin{bmatrix}B_c & G_c\end{bmatrix}\begin{bmatrix}D^T D & 0\\ 0 & -\mu^2 I_l\end{bmatrix}^{-1}\begin{bmatrix}B^T\\ G^T\end{bmatrix}C^T\right)^{-1}\times$$

$$C\begin{bmatrix}B_c & G_c\end{bmatrix}\begin{bmatrix}D^T D & 0\\ 0 & -\mu^2 I_l\end{bmatrix}^{-1}\begin{bmatrix}B^T\\ G^T\end{bmatrix}.$$

To simplify the latter expression; let us substitute $I_p+C\begin{bmatrix}B_c & G_c\end{bmatrix}\begin{bmatrix}D^T D & 0\\ 0 & -\mu^2 I_l\end{bmatrix}^{-1}\begin{bmatrix}B^T\\ G^T\end{bmatrix}C^T$ by $\mathbb{R}$.

Hence, S_c is reduced into the form

$$
\begin{aligned}
S_c = {} & 2B_c D^T D B_c^T C^T \mathbb{R}^{-1} C B_c D^T D B_c^T \\
& + 2B_c D^T D B_c^T C^T \mathbb{R}^{-1} C \tfrac{G_c G_c^T}{\mu^2} + 2B_c D^T D B_c^T C^T \\
& - 2\tfrac{G_c G_c^T}{\mu^2} - 2\tfrac{G_c G_c^T}{\mu^2} C^T \mathbb{R}^{-1} C \tfrac{G_c G_c^T}{\mu^2} \\
= {} & 2B_c\left(\left(D^T D\right)^{-1} - \left(D^T D\right)^{-1} B_c^T C^T \mathbb{R}^{-1} C B_c \left(D^T D\right)^{-1}\right) B_c^T \\
& - 2G_c\left(\tfrac{I_l}{\mu^2} + \tfrac{G_c^T}{\mu^2} C^T \mathbb{R}^{-1} C \tfrac{G_c}{\mu^2}\right) G_c^T \\
& + 2B_c\left(D^T D\right)^{-1} B_c^T C^T \mathbb{R}^{-1} C \tfrac{G_c G_c^T}{\mu^2} \\
& + 2\tfrac{G_c G_c^T}{\mu^2} C^T \mathbb{R}^{-1} C B_c \left(D^T D\right)^{-1} B_c^T
\end{aligned}
$$

In order to compute Z_c and R_c, we introduce a matrix $\boldsymbol{L}$ such that the latter expression of S_c becomes

$$
S_c = 2\left(B_c + G_c \boldsymbol{L}\right)\left(\left(D^T D\right)^{-1} - \left(D^T D\right)^{-1} B_c^T C^T \mathbb{R}^{-1} C B_c \times \left(D^T D\right)^{-1}\right)\left(B_c + G_c \boldsymbol{L}\right)^T
$$

with $\boldsymbol{L} = \frac{G_c^T}{\mu^2} C^T \mathbb{R}^{-1} C B_c \left(I_m - \left(D^T D\right)^{-1} B_c^T C^T \mathbb{R}^{-1} C B_c\right)^{-1}$.

Then by using the identity :$(I + \delta\beta)^{-1}\delta = \delta(I + \beta\delta)^{-1}$ for$\delta = \mathbb{R}^{-1} C B_c$ and $\beta = -\left(D^T D\right)^{-1} B_c^T C^T$ and the matrix $\boldsymbol{L}$ is reduced into $\boldsymbol{L} = \frac{G_c^T}{\mu^2} C^T \left(I_p - C\frac{G_c G_c^T}{\mu^2} C^T\right) C B_c$.

And Z_c is reduced into $Z_c = \frac{2}{\mu^2} G_c \left(I_l - \frac{G_c^T C^T C G_c}{\mu^2}\right) G_c^T$ by considering the fact that $\left(I_p - C\frac{G_c G_c^T}{\mu^2} C^T\right)^{-1} =$ $\mathbb{R}^{-1} + \mathbb{R}^{-1} C B_c \left(D^T D\right)^{-1} B_c^T C^T \left(I_p - C\frac{G_c G_c^T}{\mu^2} C^T\right)^{-1}$.

The matrix $\boldsymbol{B} = B_c + G_c \boldsymbol{L}$ is computed as

$$
\begin{aligned}
\boldsymbol{B} & = \left(I_n + \tfrac{G_c G_c^T}{\mu^2} C^T \left(I_p - C\tfrac{G_c G_c^T}{\mu^2} C^T\right)^{-1} C\right) B_c \\
& = \left(I_n + \tfrac{G_c G_c^T}{\mu^2} C^T C \left(I_n - \tfrac{G_c G_c^T}{\mu^2} C^T C\right)^{-1}\right) B_c .
\end{aligned}
$$

Henceforth $\boldsymbol{B} = \left(I_n - \frac{G_c G_c^T}{\mu^2} C^T C\right)^{-1} B_c$.

And the matrix R_c is

$$
\begin{aligned}
R_c & = \tfrac{1}{2}\left(\left(D^T D\right)^{-1} - \left(D^T D\right)^{-1} B_c^T C^T \mathbb{R}^{-1} C B_c \left(D^T D\right)^{-1}\right)^{-1} \\
& = \tfrac{1}{2} D^T D + \tfrac{1}{2} B_c^T C^T \left(I_p - C\tfrac{G_c G_c^T}{\mu^2} C^T\right)^{-1} C B_c
\end{aligned}
$$

and then by applying :$(I + \delta\beta)^{-1}\delta = \delta(I + \beta\delta)^{-1}$ for $\delta = C^T$ and $\beta = -C\frac{G_c G_c^T}{\mu^2}$, we obtain in the end

$$
R_c = \tfrac{1}{2} D^T D + \tfrac{1}{2} B_c^T \left(I_n - \tfrac{G_c G_c^T}{\mu^2} C^T C\right)^{-1} C^T C B_c \blacksquare
$$