OPTIMAL POLICIES FOR A CONGESTED URBAN NETWORK

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ABSTRACT

In this paper, we focus on car traffic simulation and optimization of a portion of the Salerno urban network in Italy. The car densities evolution is described by a fluid dynamic model. A cost functional, that measures the kinetic energy on roads, is maximized using a decentralized approach. In particular, to improve viability conditions, we apply locally optimal distribution coefficients at each 1×3 junction (one incoming road and three outgoing roads) and optimal right of way parameters at each 2×1 junctions (two incoming roads and one outgoing road). The goodness of the optimization results has been confirmed by simulations.

Keywords: conservation laws, simulation, optimization, decongestion.

1. INTRODUCTION

Urban infrastructures, as consequence of the increasing number of vehicles on road networks, are frequently characterized by a high car density, with possible birth of congestions, that cause unwished effects. The most typical ones are a reduction of cars velocities, queues formations and backward propagations, impossibility for drivers to forecast the travel times, pollution problems due to fuel consumptions. In worst cases, hard congestion levels can provoke car accidents and additional problems connected to the emergency situations management. In such a context, the problem of traffic modelling and control of cars flows assumes a great importance.

The aim of this paper is to discuss some optimization results obtained for a portion of Salerno urban network, in Italy, characterized by a heavy traffic, since it separates the centre of the city from outskirts. The topology of the network, reported in Figure 1, consists of eight roads. Each road is divided into segments, indicated by letters: Traversa Federico Romano (segments *a*, *b*, *c*), Via Parmenide (segment *d*), Via Picenza (segment *e*), Via Fiume (segment *f*), Piazza Monsignor Grasso (segment *g*), Via Trento (segments *h*, and *i*), Traversa Giuseppe Olivieri (segment *l*) and Via Davide Galdi (segment *m*). In particular, *c*, *g*, and *h* are inner road segments, while *a*, *b*, *d*, *e*, *f*, *i*, *l*, *m* external ones. Road junctions are labelled by numbers: 1 and 2 are of 2×1 type (two incoming roads and one outgoing road), 3 and 4 of 1×3 type (one incoming road and three outgoing roads).

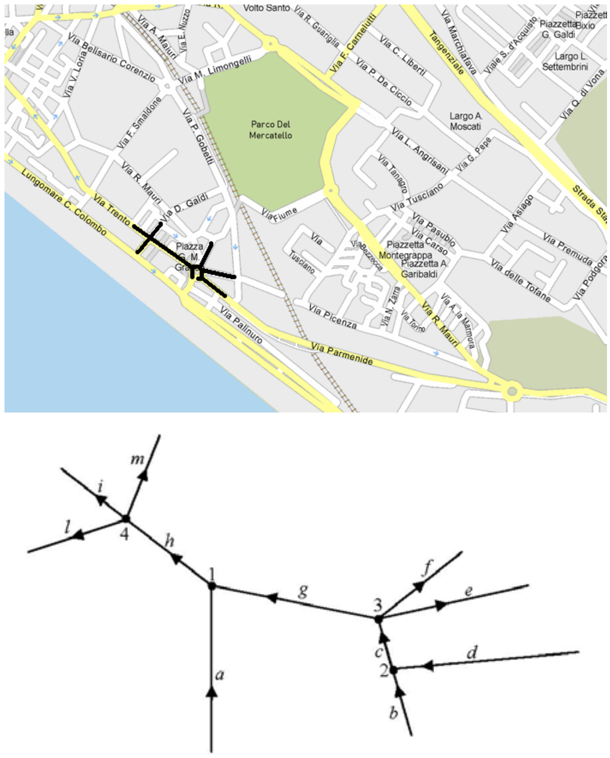

Figure 1: a portion of the real network of Salerno and its graph

For the description of traffic flows, we follow a fluid dynamic approach: the evolution of car densities is given on each road by a conservation law (Lighthill et al. 1955, Richards 1956). Dynamics at junctions is uniquely solved (Coclite et al. 2005, Garavello et al. 2006) adopting rules for the traffic distribution at nodes, the flux maximization, and the right of ways (if the number of incoming roads at nodes is greater than that of outgoing ones). Observe that the traffic evolution on the case study is captured using right of way parameters, which discriminate among priorities of incoming roads flows, in the case of 2×1 junctions, and distribution coefficients which describe the amount of traffic that, from incoming roads, is directed to outgoing ones, in the case of 1×3 junctions.

In order to improve the viability conditions, we define a cost functional, *E*, that measures the kinetic energy of drivers on roads of the considered network. Here, we want to maximize E with respect to distribution coefficients at junctions 1×3 and right of way parameters at junctions 2×1. Some control problems have already been considered for traffic parameters of fluid dynamic models, see Gugat et al. 2005, Helbing et al. 2005 and Herty et al. 2003. In particular, in Cascone et al. 2007, Cascone et al. 2008 and Cutolo et al. 2009, different cost functionals, measuring average velocity, average travelling time, weighted and not weighted with the number of cars, flux, and kinetic energy, have been introduced and optimized for 1×2 and 2×1 junctions. Moreover, parameters of 2×2 junctions have been optimized for the fast transit of emergency vehicles in critical situations, such as accidents (Manzo et al. 2009).

The analysis of the cost functional E on complex networks is a very difficult problem, hence we follow a decentralized approach: for asymptotic E, optimal right of way parameters and distribution coefficients are found, respectively, for each 2×1 junction and each 1×3 one. For the first junctions type, we use the exact optimal solutions found in Cutolo et al. 2009; for the second one, never studied before, optimal distribution coefficients are numerically computed. The global (sub)optimal solutions for the examined network is then obtained by localization: the optimal solution is applied locally for each time at each junction.

The optimization results are tested by simulations (for numerics, see Bretti et al. 2006, Godlewsky et al. 1996, Godunov 1959, Lebacque 1996). Two different choices of distribution coefficients and right of way parameters are analyzed: optimal values given by the optimization algorithms, and random values, i.e. at the beginning of the simulation process, a random value of traffic parameters is kept constant during all simulations.

First, we consider a simple 1×3 junction to test the goodness of the numerical optimization algorithm for distribution coefficients. Then, we examine the real case study, that shows some interesting features: when random coefficients are used, hard congestions are frequent, as expected; the use of optimal traffic parameters allows a local redistribution of traffic flows at 1×3 junctions and reduction of traffic densities at 2×1 junctions, with consequent benefits in terms of roads decongestions, on the global performance of the network.

The paper is organized as follows. In Section 2, we introduce the model. Section 3 is devoted to the solutions of Riemann Problems at 2×1 and 1×3 road junctions. The cost functional, measuring the kinetic energy of cars, is presented in Section 4, where we discuss its optimization. Simulations for a single junction and for the case study are presented in Section 5. The paper ends with conclusions.

2. ROAD NETWORKS MODEL

A road network is described by a couple $(\mathtt{I},\mathtt{J})$, where $\mathtt{I}$ is the set of roads, modelled by intervals $[a_i,b_i]\subset R$, $i=1,...,N$ and $\mathtt{J}$ the set of junctions, which connect roads.

On each road, the traffic evolution is given by the conservation law (Lighthill et al. 1955, Richards 1956):

$$\rho_t + f(\rho)_x = 0, \tag{1}$$

where $\rho=\rho(t,x)\in[0,\rho_{\max}]$ is the car density, $\rho_{\max}$ is the maximal density, $f(\rho)=\rho v(\rho)$ the flux with $v(\rho)$ average velocity.

Considering $\rho_{\max}=1$ and a decreasing velocity $v(\rho)=1-\rho$, $\rho\in[0,1]$, we get the flux:

$$f(\rho)=\rho(1-\rho),\ \rho\in[0,1], \tag{2}$$

which is a strictly concave C^2 function such that $f(0)=f(1)=0$, with a unique maximum $\sigma=\frac{1}{2}$.

At junctions, traffic dynamics is found solving Riemann Problems, i.e. Cauchy Problems with a constant initial datum for each incoming and outgoing road.

Fix a junction J of $n\times m$ type (n incoming roads I_φ, $\varphi=1,...,n$, and m outgoing roads, I_ψ, $\psi=n+1,...,n+m$), and an initial datum $\rho_0=(\rho_{1,0},...,\rho_{n,0},\rho_{n+1,0},...,\rho_{n+m,0})$.

Definition. *A Riemann Solver (RS) for the junction* J *is a map* $RS:[0,1]^n\times[0,1]^m\to[0,1]^n\times[0,1]^m$ *that associates to Riemann data* $\rho_0=(\rho_{\varphi,0},\rho_{\psi,0})$ *at* J *a vector* $\hat{\rho}=(\hat{\rho}_\varphi,\hat{\rho}_\psi)$ *so that the solution on an incoming road* I_φ, $\varphi=1,...,n$, *is the wave* $(\rho_{\varphi,0},\hat{\rho}_\varphi)$ *and on an outgoing one* I_ψ, $\psi=n+1,...,n+m$, *is the wave* $(\hat{\rho}_\psi,\rho_{\psi,0})$. *We require that the following conditions hold:*

(c1) $RS(RS(\rho_0))=RS(\rho_0)$;

(c2) *on each incoming road* I_φ, $\varphi=1,...,n$, *the wave* $(\rho_{\varphi,0},\hat{\rho}_\varphi)$ *has negative speed, while on each outgoing road* I_ψ, $\psi=n+1,...,n+m$, *the wave* $(\hat{\rho}_\psi,\rho_{\psi,0})$ *has positive speed.*

If $m\geq n$, a possible RS at J can be defined by the following rules (see Coclite et al. 2005):

(A) traffic is distributed at J according to some coefficients, collected in a traffic distribution matrix $A = (\alpha_{j,i})$, $0 < \alpha_{j,i} < 1$, $i \in \{1,...,n\}$, $j \in \{n+1,...,n+m\}$, $\sum_{j=n+1}^{n+m} \alpha_{j,i} = 1$. The i–th column of A indicates the percentages of traffic that, from the incoming road I_i, distribute to the outgoing roads;

(B) respecting (A), drivers maximize the flux through J.

Otherwise, if $m < n$, we need the additional rule (C), beside (A) and (B):

(C) Assume that not all cars can enter the outgoing roads, and let C be the amount that can do it. Then, $p_i C$ cars come from the incoming road i, where $p_i \in]0,1[$ is the right of way parameter of road i, $i = 1,...,n$, and $\sum_{i=1}^{n} p_i = 1$.

Assuming an initial datum $\rho_0 = (\rho_{\varphi,0}, \rho_{\psi,0})$ and the flux function (2), the solution $\hat{\rho}$ of the RS at J is given by:

$$\hat{\rho}_\varphi \in \begin{cases} \{\rho_{\varphi,0}\} \cup]\tau(\rho_{\varphi,0}),1], & \text{if } 0 \le \rho_{\varphi,0} < \frac{1}{2}, \\ \left[\frac{1}{2},1\right], & \text{if } \frac{1}{2} \le \rho_{\varphi,0} \le 1, \end{cases} \tag{3}$$

$\varphi = 1,...,n$, and

$$\hat{\rho}_\psi \in \begin{cases} [0,\frac{1}{2}], & \text{if } 0 \le \rho_{\psi,0} \le \frac{1}{2}, \\ \{\rho_{\psi,0}\} \cup [0,\tau(\rho_{\psi,0})[, & \text{if } \frac{1}{2} < \rho_{\psi,0} \le 1, \end{cases} \tag{4}$$

$\psi = n+1,...,n+m,$

where $\tau : [0,1] \to [0,1]$ is the map such that $f(\rho) = f(\tau(\rho))$ and $\tau(\rho) \ne \rho$ if $\rho \ne \frac{1}{2}$.

3. RIEMANN SOLVERS

In this section, we consider the flux function (2) and describe the construction of Riemann Solvers at junctions, which satisfies rules (A), (B) and (C).

Fix a junction J of $n \times m$ type. We indicate the cars densities on incoming roads and outgoing ones, respectively, by:

$$\rho_\varphi(t,x) \in [0,1], \ (t,x) \in R^+ \times I_\varphi, \quad \varphi = 1,...,n, \tag{5}$$

$$\rho_\psi(t,x) \in [0,1], \ (t,x) \in R^+ \times I_\psi, \quad \psi = n+1,...,n+m. \tag{6}$$

From condition (c2), fixing the flux function (2) and assuming $\rho_0 = (\rho_{1,0},...,\rho_{n,0},\rho_{n+1,0},...,\rho_{n+m,0})$ as the initial datum of an RP at J, the maximum fluxes on roads are:

$$\gamma_\varphi^{\max} = f(\rho_{\varphi,0}) H\left(\tfrac{1}{2} - \rho_{\varphi,0}\right) + f\left(\tfrac{1}{2}\right) H\left(\rho_{\varphi,0} - \tfrac{1}{2}\right), \quad \varphi = 1,...,n, \tag{7}$$

$$\gamma_\psi^{\max} = f\left(\tfrac{1}{2}\right) H\left(\tfrac{1}{2} - \rho_{\psi,0}\right) + f(\rho_{\psi,0}) H\left(\rho_{\psi,0} - \tfrac{1}{2}\right), \quad \psi = n+1,...,n+m, \tag{8}$$

where $H(\bullet)$ is the Heavyside function.

According to the real case study, we analyze RSs for two junction types: 2×1 and 1×3.

3.1. The case *n* = 2 and *m* = 1

Consider a junction J of 2×1 type (two incoming roads, 1 and 2, and one outgoing road, 3).

The solution to the RP at J with initial datum $\rho_0 = (\rho_{1,0}, \rho_{2,0}, \rho_{3,0})$ is constructed in the following way. Since, from rule (B), the aim is to maximize the through traffic, we set:

$$\hat{\gamma}_3 = \min\left\{\gamma_1^{\max} + \gamma_2^{\max}, \gamma_3^{\max}\right\}. \tag{9}$$

Introduce the conditions:

(A1) $p\gamma_3^{\max} < \gamma_1^{\max}$;

(A2) $(1-p)\gamma_3^{\max} < \gamma_2^{\max}$.

If $\hat{\gamma}_3 = \gamma_1^{\max} + \gamma_2^{\max}$, the solution to the RP is $\hat{\gamma} = (\gamma_1^{\max}, \gamma_2^{\max}, \gamma_1^{\max} + \gamma_2^{\max})$.

If $\hat{\gamma}_3 = \gamma_3^{\max}$, we have that:

- $\hat{\gamma} = (p\gamma_3^{\max}, (1-p)\gamma_3^{\max}, \gamma_3^{\max})$ when A1 and A2 are both satisfied;
- $\hat{\gamma} = (\gamma_3^{\max} - \gamma_2^{\max}, \gamma_2^{\max}, \gamma_3^{\max})$ when A1 holds and A2 is not satisfied;
- $\hat{\gamma} = (\gamma_1^{\max}, \gamma_3^{\max} - \gamma_1^{\max}, \gamma_3^{\max})$ when A1 is not satisfied and A2 holds.

The case of both A1 and A2 false is not possible, since it would be $\gamma_3^{\max} > \gamma_1^{\max} + \gamma_2^{\max}$.

Once $\hat{\gamma}$ is known, from (3) and (4) we get the solution $\hat{\rho}$.

3.2. The case *n* = 1 and *m* = 3

Fix a junction J of 1×3 type (one incoming road, 1, and three outgoing roads, 2, 3, and 4).

For $\psi = 2,3,4$, if $\alpha_{\psi-1} \in \left]0,1\right[$, $\sum_{\psi=2}^{4} \alpha_{\psi-1} = 1$, represents the percentage of cars, which, from road 1, goes to road ψ, the fluxes solution to the RP at J are:

$$\hat{\gamma} = (\hat{\gamma}_1, \alpha_1\hat{\gamma}_1, \alpha_2\hat{\gamma}_1, \alpha_3\hat{\gamma}_1), \tag{10}$$

where

$$\hat{\gamma}_1 = \min\left\{\gamma_1^{\max}, \frac{\gamma_2^{\max}}{\alpha_1}, \frac{\gamma_3^{\max}}{\alpha_2}, \frac{\gamma_4^{\max}}{\alpha_3}\right\}, \tag{11}$$

with $\alpha_3 = 1 - \alpha_1 - \alpha_2$.

Notice that $\hat{\gamma}_1$ is dependent on values of α_1 and α_2. Define the variable:

$$\zeta_{i,j} = \frac{\gamma_i^{\max}}{\gamma_j^{\max}},\ i \neq j,\ i = 1,\dots,4,\ j = 1,\dots,4, \tag{12}$$

and the following sets:

$$\Omega_1 = \left\{\begin{aligned} &(\alpha_1,\alpha_2) \in R^2 : \alpha_1 \le \zeta_{2,1}, \\ &\alpha_2 \le \zeta_{3,1},\ \alpha_1 + \alpha_2 \ge 1 - \zeta_{4,1} \end{aligned}\right\}, \tag{13}$$

$$\Omega_2 = \left\{\begin{aligned} &(\alpha_1,\alpha_2) \in R^2 : \alpha_1 \ge \zeta_{2,1}, \\ &\alpha_2 \le \alpha_1\zeta_{3,1},\ \alpha_2 \ge 1 - \alpha_1\left(1 + \zeta_{4,2}\right) \end{aligned}\right\}, \tag{14}$$

$$\Omega_3 = \left\{\begin{aligned} &(\alpha_1,\alpha_2) \in R^2 : \alpha_2 \ge \zeta_{3,1},\ \alpha_2 \ge \alpha_1\zeta_{3,2}, \\ &\alpha_2 \ge -\alpha_1\frac{1}{1+\zeta_{4,3}} + \frac{1}{1+\zeta_{4,3}} \end{aligned}\right\}, \tag{15}$$

$$\Omega_4 = \left\{\begin{aligned} &(\alpha_1,\alpha_2) \in R^2 : \alpha_1 + \alpha_2 \le 1 - \zeta_{4,1}, \\ &\alpha_2 \le 1 - \alpha_1\left(1 + \zeta_{4,2}\right), \alpha_2 \le 1 - \alpha_1\left(1 + \zeta_{3,2}\right) \end{aligned}\right\}. \tag{16}$$

The open set

$$\Omega = \left\{\begin{aligned} &(\alpha_1,\alpha_2) \in R^2 : 0 < \alpha_1 < 1, \\ &0 < \alpha_2 < 1,\ \alpha_1 + \alpha_2 < 1 \end{aligned}\right\} \tag{17}$$

is decomposed as $\Omega = \bigcup_{k=1}^{4} \Lambda_i$, where $\Lambda_1 = \Omega \cap \Omega_1$, $\Lambda_2 = \Omega \cap \Omega_2$, $\Lambda_3 = \Omega \cap \Omega_3$, and $\Lambda_4 = \Omega \cap \Omega_4$.

A unique RS is associated to each region Λ_i, $i = 1,\dots,4$. Precisely, we have that:

if $(\alpha_1,\alpha_2) \in \Lambda_1$, $\hat{\gamma} = \left(\gamma_1^{\max}, \alpha_1\gamma_1^{\max}, \alpha_2\gamma_1^{\max}, \alpha_3\gamma_1^{\max}\right)$,

if $(\alpha_1,\alpha_2) \in \Lambda_2$, $\hat{\gamma} = \left(\frac{\gamma_2^{\max}}{\alpha_1}, \gamma_2^{\max}, \alpha_2\frac{\gamma_2^{\max}}{\alpha_1}, \alpha_3\frac{\gamma_2^{\max}}{\alpha_1}\right)$,

if $(\alpha_1,\alpha_2) \in \Lambda_3$, $\hat{\gamma} = \left(\frac{\gamma_3^{\max}}{\alpha_2}, \alpha_1\frac{\gamma_3^{\max}}{\alpha_2}, \gamma_3^{\max}, \alpha_3\frac{\gamma_3^{\max}}{\alpha_2}\right)$,

if $(\alpha_1,\alpha_2) \in \Lambda_4$, $\hat{\gamma} = \left(\frac{\gamma_4^{\max}}{\alpha_3}, \alpha_1\frac{\gamma_4^{\max}}{\alpha_3}, \alpha_2\frac{\gamma_4^{\max}}{\alpha_3}, \gamma_4^{\max}\right)$.

Once $\hat{\gamma}$ is known, the solution of the RS $\hat{\rho}$ is easily found again from (3) and (4).

3.3. Examples

Consider a junction J of 2×1 type, an initial datum $\rho_0 = (0.25, 0.15, 0.3)$ and $p = 0.9$. From (7) and (8), the maximal fluxes on roads are:

$$\gamma_1^{\max} = 0.1875,\ \gamma_2^{\max} = 0.1275,\ \gamma_3^{\max} = 0.25. \tag{18}$$

Hence, we have that:

$$\hat{\gamma}_3 = \min\left\{\gamma_1^{\max} + \gamma_2^{\max}, \gamma_3^{\max}\right\} = 0.25,$$

and

$$\begin{aligned} \hat{\gamma} &= \left(\gamma_1^{\max}, \gamma_3^{\max} - \gamma_1^{\max}, \gamma_3^{\max}\right) = \\ &= (0.1875, 0.0625, 0.25). \end{aligned} \tag{19}$$

From (3) and (4), the density solutions are:

$$\hat{\rho} = (0.25, 0.933013, 0.5). \tag{20}$$

For a junction J of 1×3 type, assign the initial datum $\rho_0 = (0.4, 0.95, 0.75, 0.85)$. We get that:

$$\begin{aligned} &\gamma_1^{\max} = 0.24,\ \gamma_2^{\max} = 0.0475; \\ &\gamma_3^{\max} = 0.1875,\ \gamma_4^{\max} = 0.1275. \end{aligned} \tag{21}$$

Regions Λ_i are depicted in Figure 2. Assume $\alpha_1 = \alpha_2 = 0.3$, $(\alpha_1,\alpha_2) \in \Lambda_2$, hence:

$$\begin{aligned} \hat{\gamma} &= \left(\frac{\gamma_2^{\max}}{\alpha_1}, \gamma_2^{\max}, \alpha_2\frac{\gamma_2^{\max}}{\alpha_1}, \alpha_3\frac{\gamma_2^{\max}}{\alpha_1}\right) = \\ &= (0.1583, 0.0475, 0.0475, 0.0633). \end{aligned} \tag{22}$$

The corresponding density solutions are:

$$\hat{\rho} = (0.802, 0.95, 0.05, 0.068). \tag{23}$$

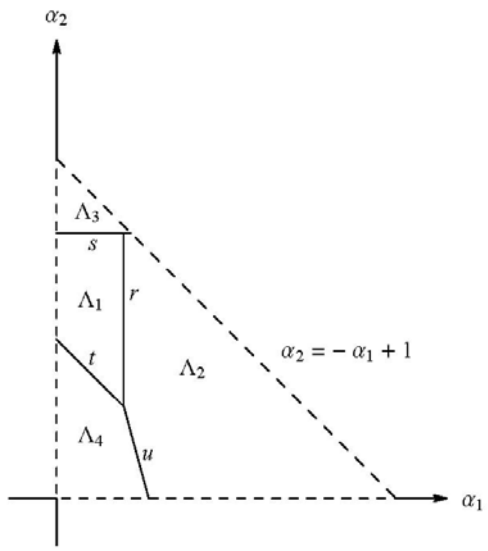

Figure 2: Decomposition of the region Ω, where $r : \alpha_1 = \zeta_{2,1}$, $s : \alpha_2 = \zeta_{3,1}$, $t : \alpha_2 = 1 - \zeta_{4,1} - \alpha_1$, and $u : \alpha_2 = 1 - \alpha_1 (1 + \zeta_{4,2})$

4. OPTIMIZATION OF ROAD NETWORKS

Consider a junction J of type $n \times m$ and an initial datum $\rho_0 = (\rho_{\varphi,0}, \rho_{\psi,0})$. We define the cost functional $E(t)$ as:

$$E(t) = \sum_{\varphi=1}^{n} \int_{I_\varphi} f(\rho_\varphi(t,x)) v(\rho_\varphi(t,x)) dx + \sum_{\psi=n+1}^{n+m} \int_{I_\psi} f(\rho_\psi(t,x)) v(\rho_\psi(t,x)) dx, \tag{24}$$

which measures the kinetic energy of cars travelling at the junction.

Assigned a time horizon $[0,T]$, with T sufficiently big, we want to maximize $\int_0^T E(t) dt$ by a suitable choice of traffic coefficients $\alpha_{\psi,\varphi} \in]0,1[$ or right of way parameters (if $n > m$) $p_\varphi \in]0,1[$, $\varphi = 1, ..., n$, $\psi = n+1, ..., n+m$.

4.1. The case *n* = 2 and *m* = 1

Given a junction J of 2×1 type, for the flux function (2) and T sufficiently big, the cost functional $E(T)$ assumes the form:

$$E(T) = \frac{1}{2} \sum_{i=1}^{3} \hat{\gamma}_i \left(1 - s_i \sqrt{1 - 4\hat{\gamma}_i}\right) \tag{25}$$

where, for $\varphi = 1, 2$:

$s_\varphi = -1$ if $\rho_{\varphi,0} < \frac{1}{2}$ and $\gamma_1^{\max} + \gamma_2^{\max} \le \gamma_3^{\max}$ or $\rho_{\varphi,0} < \frac{1}{2}$, $\gamma_3^{\max} < \gamma_1^{\max} + \gamma_2^{\max}$ and $p_\varphi \hat{\gamma}_3 > \gamma_\varphi^{\max}$;

$s_\varphi = +1$ if $\rho_{\varphi,0} \ge \frac{1}{2}$, or $\rho_{\varphi,0} < \frac{1}{2}$, $\gamma_3^{\max} < \gamma_1^{\max} + \gamma_2^{\max}$ and $p_\varphi \hat{\gamma}_3 < \gamma_\varphi^{\max}$;

$s_3 = -1$ if $\rho_{3,0} \le \frac{1}{2}$ or $\rho_{3,0} > \frac{1}{2}$, $\gamma_1^{\max} + \gamma_2^{\max} < \gamma_3^{\max}$;

$s_3 = +1$ if $\rho_{3,0} > \frac{1}{2}$ and $\gamma_1^{\max} + \gamma_2^{\max} \ge \gamma_3^{\max}$,

with

$$p_\varphi = \begin{cases} p, & \text{if } \varphi = 1, \\ 1 - p, & \text{if } \varphi = 2. \end{cases} \tag{26}$$

The optimal choice of right of way parameters is found analytically according to Cutolo et al. 2009 as follows.

Theorem. *Consider a junction J of 2×1 type. For the flux function (2) and T sufficiently big, $E(t)$ is optimized for the following values of p:*

1) *if $s_1 = s_2 = +1$, then:*
   (a) $p = \frac{1}{2}$, *if* $\beta^- \le 1 \le \beta^+$ *or* $\gamma_2^{\max} = \gamma_3^{\max}$;
   (b) $p \in \left]0, p^-\right]$, *if* $\beta^- \le \beta^+ \le 1$;
   (c) $p \in \left[p^+, 1\right[$, *if* $1 \le \beta^- \le \beta^+$;
2) *if $s_1 = -1 = -s_2$, then:*
   (a) $p = \frac{1}{2}$ *or* $p \in \left[p^+, 1\right[$, *if* $\beta^- \le 1 \le \beta^+$ *or* $\gamma_2^{\max} = \gamma_3^{\max}$;
   (b) $p \in \left]0, p^-\right]$ *or* $p \in \left[p^+, 1\right[$, *if* $\beta^- \le \beta^+ \le 1$;
   (c) $p \in \left[p^+, 1\right[$, *if* $1 \le \beta^- \le \beta^+$;
3) *if $s_1 = +1 = -s_2$, then:*
   (a) $p = \frac{1}{2}$ *or* $p \in \left]0, p^-\right]$, *if* $\beta^- \le 1 \le \beta^+$;
   (b) $p \in \left]0, p^-\right]$, *if* $\beta^- \le \beta^+ \le 1$;
   (c) $p \in \left]0, p^-\right]$ *or* $p \in \left[p^+, 1\right[$, *if* $1 \le \beta^- \le \beta^+$;
   (d) $p = \frac{1}{2}$ *or* $p \in \left[p^+, 1\right[$, *if* $\gamma_2^{\max} = \gamma_3^{\max}$;
4) *if $s_1 = s_2 = -1$, then:*

(a) $p = \frac{1}{2}$ *or* $p \in \left[p^+, 1\right[$ *, if* $\beta^- \le 1 \le \beta^+$ *, with* $\beta^- \beta^+ > 1$ *or* $\gamma_2^{\max} = \gamma_3^{\max}$ *;*

(b) $p = \frac{1}{2}$ *or* $p \in \left]0, p^-\right]$ *, if* $\beta^- \le 1 \le \beta^+$ *, with* $\beta^- \beta^+ < 1$ *;*

(c) $p = \frac{1}{2}$ *or* $p \in \left]0, p^-\right] \cup \left[p^+, 1\right[$ *, if* $\beta^- \le 1 \le \beta^+$ *, with* $\beta^- \beta^+ = 1$ *;*

(d) $p \in \left]0, p^-\right]$ *, if* $\beta^- \le \beta^+ \le 1$ *;*

(e) $p \in \left[p^+, 1\right[$ *, if* $1 \le \beta^- \le \beta^+$ *,*

where

$$p^- = 1 - \zeta_{2,3},\ p^+ = \zeta_{1,3},$$
$$\beta^- = \xi_{3,1} - 1,\ \beta^+ = \frac{1}{\zeta_{3,2} - 1}. \tag{27}$$

4.2. The case *n* = 1 and *m* = 3

For a junction J of 1×3 type, assuming T sufficiently big, the cost functional $E(T)$ becomes:

$$E(T) = \frac{\hat{\gamma}_1 \left(1 - s_1 \sqrt{1 - 4\hat{\gamma}_i}\right)}{2} + \frac{\hat{\gamma}_1}{2} \sum_{\psi=2}^{4} \alpha_{\psi-1} \left(1 - s_\psi \sqrt{1 - 4\alpha_{\psi-1}\hat{\gamma}_1}\right) \tag{28}$$

where

$s_1 = +1$ if $\rho_{1,0} \ge \frac{1}{2}$, or $\rho_{1,0} < \frac{1}{2}$ and $\hat{\gamma}_1 > \min\left\{\frac{\gamma_2^{\max}}{\alpha_1}, \frac{\gamma_3^{\max}}{\alpha_2}, \frac{\gamma_4^{\max}}{\alpha_3}\right\}$;

$s_1 = -1$ if $\rho_{1,0} < \frac{1}{2}$ and $\hat{\gamma}_1 \le \min\left\{\frac{\gamma_2^{\max}}{\alpha_1}, \frac{\gamma_3^{\max}}{\alpha_2}, \frac{\gamma_4^{\max}}{\alpha_3}\right\}$;

for $\psi = 2,3,4$:

$s_\psi = +1$ if $\rho_{\psi,0} > \frac{1}{2}$ and, for $\psi' \ne \psi$, $\psi'' \ne \psi' \ne \psi$, $\frac{\gamma_\psi^{\max}}{\alpha_{\psi-1}} \le \min\left\{\gamma_1^{\max}, \frac{\gamma_{\psi'}^{\max}}{\alpha_{\psi'-1}}, \frac{\gamma_{\psi''}^{\max}}{\alpha_{\psi''-1}}\right\}$;

$s_\psi = -1$ if $\rho_{\psi,0} \le \frac{1}{2}$ or, for $\psi' \ne \psi$, $\psi'' \ne \psi' \ne \psi$, $\rho_{\psi,0} > \frac{1}{2}$ and $\frac{\gamma_\psi^{\max}}{\alpha_{\psi-1}} > \min\left\{\gamma_1^{\max}, \frac{\gamma_{\psi'}^{\max}}{\alpha_{\psi'-1}}, \frac{\gamma_{\psi''}^{\max}}{\alpha_{\psi''-1}}\right\}$.

Observe that since s_i and $\hat{\gamma}_i$, $i = 1,...,4$, depend on the initial data, the functional (28) assumes different expressions in each region Λ_i, $i = 1,...,4$, and to find the analytical optimal distribution coefficients is a huge task. Hence the values of α_1 and α_2, which optimize (28), are found numerically through the software Mathematica.

Remark. *The choice of the initial datum at J strongly influences the values of optimal* α_1 *and* α_2 *.*
For example, if $(\alpha_1, \alpha_2) \in \Lambda_1$ *,* $\hat{\gamma}_1 = \gamma_1^{\max}$ *, and*

$$E(T) = \frac{\gamma_1^{\max}}{2}\left(1 - s_1\sqrt{1 - 4\gamma_1^{\max}}\right) + \frac{\gamma_1^{\max}}{2}\sum_{i=2}^{4} \alpha_{i-1}\left(1 - s_i\sqrt{1 - 4\alpha_{i-1}\gamma_1^{\max}}\right) \tag{29}$$

The functional has an analytical maximum for $\bar{\alpha}_1 = \bar{\alpha}_2 = \frac{1}{3}$*, which is the optimal solution only if* $(\bar{\alpha}_1, \bar{\alpha}_2) \in \Lambda_1$*. Otherwise, some numerical methods are needed.*

5. SIMULATIONS

In this section, we present some simulation results in order to test the numerical algorithm for the optimization of 1×3 junctions and to analyse the effects of random and optimal choices of distribution coefficients and right of way parameters on the real case study.

5.1. Single junctions

Consider a junction of 1×3 type. Again the incoming road is labelled with 1, while outgoing roads with 2, 3, and 4. We compare different behaviours of the cost functional (28) using: optimal numerical distribution coefficients (*optimal case*); random distribution coefficients, namely parameters taken randomly when the simulation starts and then kept constant (*random case*).

The road traffic evolution is simulated using the flux function (2) in a time interval $[0, T]$, where $T = 30$ min. Numerical approximations are made by the Godunov method with space step $\Delta x = 0.0125$ and Δt given by the CFL condition (see Godunov 1959). We assume that, at the starting instant of simulation (*t* = 0), all roads of the network are empty. Moreover, we choose the following Dirichlet boundary data: $\rho_{1,b} = 0.4$, $\rho_{2,b} = 0.65$, $\rho_{3,b} = 0.75$, $\rho_{4,b} = 0.85$.

Notice that initial conditions and boundary data are such that the network dynamics exhibits congestions on outgoing roads and optimal distribution coefficients are given by $\alpha_1 = \alpha_2 = \frac{1}{3}$.

Figures 3 and 4 show that, simulating the junction with optimal α_1 and α_2 values, traffic conditions are improved with respect to random cases: the optimal cost functional is higher with respect to others. In particular, although the asymptotic state is not reached (the final

T), the optimization algorithm always gives better performances than other simulation cases.

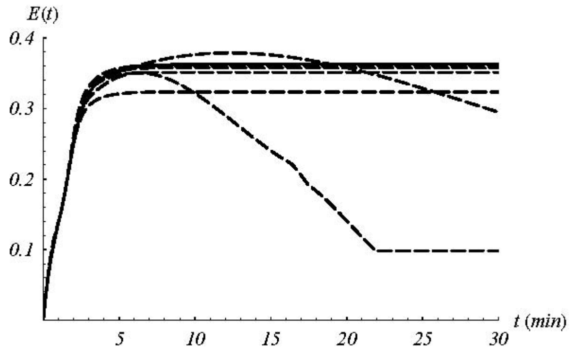

Figure 3: behaviour of $E(t)$ for optimal choice of α_1 and α_2 (solid line), and random distribution coefficients (dashed lines)

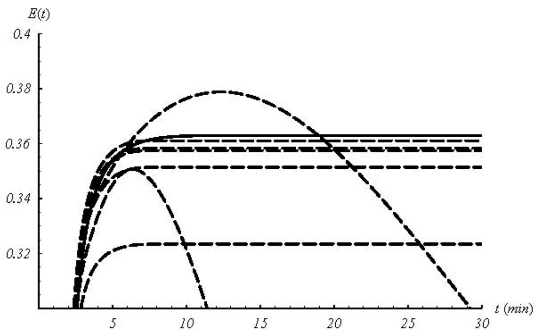

Figure 4: zoom of $E(t)$ for optimal choice of α_1 and α_2 (solid line), and random distribution coefficients (dashed lines)

5.2. Real network in Salerno

In this subsection we present some simulation results for the network of Figure 1.

The evolution of traffic flows is simulated in a time interval $[0,T]$, where $T = 100$ min, using the Godunov method with $\Delta x = 0.0125$ and $\Delta t = \Delta x / 2$. Initial conditions and boundary data for all roads are in Table 1 and are chosen so as to simulate a network with congested roads.

Again, we consider two types of simulations: optimal and random cases. Optimal right of way parameters at junctions 1 and 2 are found according to the Theorem of previous Section, while optimal distribution coefficients at junctions 3 and 4 are computed using a numerical algorithm.

Table 1: Initial conditions and boundary data for roads of the network

Road	Initial condition	Boundary data
a	0	0.3
b	0	0.3
c	0.3	/
d	0	0.4
e	0.6	0.4
f	0.7	0.9
g	0.3	/
h	0.3	/
i	0.65	0.9
l	0.75	0.9
m	0.85	0.9

In Figures 5 and 6, we report the behaviour of the cost functional $E(t)$, defined as the sum of the kinetic energy on all network roads. Notice that random simulation curves (dashed lines) are always lower than the optimal one (continuous line). Such phenomenon is easy justified: when, at road junctions of 1×3 and 2×1 types, optimal distribution coefficients and right of way parameters are, respectively, used, traffic flows are redistributed, hence allowing a congestion reduction.

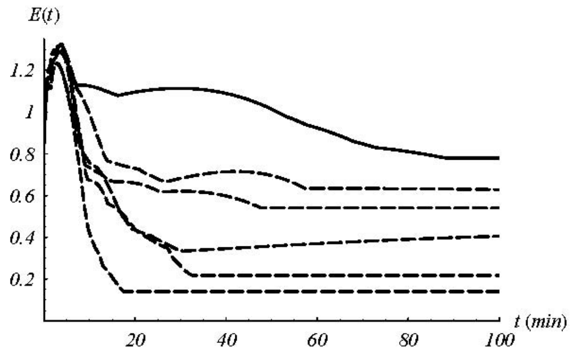

Figure 5: $E(t)$ for optimal choice of distribution coefficients (solid line) and random parameters (dashed lines)

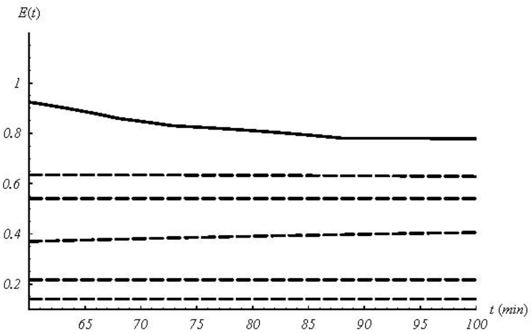

Figure 6: zoom of $E(t)$; optimal choice of distribution coefficients (solid line); random parameters (dashed lines)

6. CONCLUSIONS

In this paper, we have studied traffic flows on a portion of the Salerno urban network, in Italy.

Exact solutions for optimal right of way parameters in case of 2×1 junctions, and numerical approximations for optimal distribution coefficients in case of 1×3 junctions have been used in order to improve the viability conditions. In particular, the optimization of distribution coefficients for 1×3 junctions has been treated here for the first time.

The goodness of the optimization results has been confirmed by simulations, concluding that real benefits on traffic performances are possible.

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