DESCRIPTION AND OPTIMIZATION OF THE STRUCTURE OF HORIZIONTALLY HOMOGENEOUS PARALLEL AND DISTRIBUTED PROCESSING SYSTEMS

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ABSTRACT

A method of description and optimization of the structure of horizontally homogeneous multi-level parallel and distributed processing systems is presented. The set of feasible structures for such class of systems is defined. The description of this set is constructed in terms of the graph theory. For representation, the feasible set of structures, a condition for adjacency matrixes of adjacent levels is derived. For the reduced statement, two types of variable parameters are defined: for the level size and for the relations of adjacent levels. The formalism considered here, enables to state the structure optimization problem as a two-phase mutually dependent discrete optimization problem and to construct some classes of effective solution methods. Modelling and optimization of the structure of multi-level processing system illustrates the considered approach.

Keywords: Parallel and distributed simulation; mathematical programming; multi-level processing; multi-level selection procedure; ordering of non-ordered sets.

1. INTRODUCTION

Large-scale problems can be decomposed in many different ways (Mesarovic, Macko and Takahara 1970). The current approach for describing and optimizing the structure of hierarchical systems is based on a multi-level partitioning of given finite set in which the qualities of the system may depend on the partitioning (Riismaa, Randvee and Vain 2003). Examples of problems of this class are aggregation problems, structuring of decision-making systems, database structuring, multiple distribution or centralization problems, multi-level tournament systems, multi-level distribution systems and optimal clustering problems.

In a multi-level distribution system each element is a supplier for some lower level elements and a customer for one higher-level element. The zero-level elements are only customers and the unique top-level element is only a supplier. The choice of optimal number of suppliers-customers on each level is a mathematically complicated problem.

The multi-level tournament system (Laslier 1997) is a relatively simple special case of a multi-level processing system. To consider a tournament system, the number of games (pair-wise comparisons) is a quadratic function of the number of participants. This is a very quickly increasing function. If the number of participants was large, the number of games is very large. This is a reason why the multi-level approach is useful for the selection of the winner. From the tournaments of the first level, the winners are distributed between the tournaments of the next level. The second level tournaments´ winners are going to the third level, until the winner is selected. Suppose the goal is to minimize the number of all games. If the price for all games is the same, the solution of the problem is well known. Each tournament has two participants and only one game is played. If the prices of games for different levels are different or constraints to the number of levels are active, a relatively complicated nonlinear integer programming problem arises.

The assembling problem as well as a broad class of design and implementation problems, such as component selection in production systems, reconfiguration of manufacturing structures, optimization of the hierarchy of decision making systems, multi-level aggregation, creation and cancellation of levels, etc. can be mathematically stated as a multi-level selection problem (Riismaa, Randvee and Vain 2003).

In this paper a method of description and optimization of the structure of multi-level parallel and distributed processing systems is presented. The set of feasible structures for such class of systems is defined. The description of this set is constructed in terms of the graph theory. For representation the feasible set of structures a condition for adjacency matrixes of adjacent levels is derived.

The general problem of optimal multi-level paralleling procedure is presented. This problem is stated as a problem of selecting the feasible structure which corresponds to the minimum of total loss.

An important special case is considered, where the connection cost between the adjacent levels is the property of the supreme level: each row of the connection cost matrices between the adjacent levels consists of equal elements. It means that for each item on the next level the connection with all items on the previous level have the same costs. For this reduced statement two types of variable parameters are defined. Free variables of the inner minimization are used to describe the connections between the adjacent levels. Free variables of the outer minimization are used for the representation of the number of elements at each level.

For horizontally homogeneous hierarchies the inner minimization problem (to find optimal connection between adjacent levels) is solved analytically.

2. THE PROBLEM OF OPTIMAL MULTI-LEVEL PARALLEL AND DISTRIBUTED PROCESSING SYSTEM

Consider all *s*-levels hierarchies, where nodes on level *i* are selected from the given nonempty and disjoint sets and all selected nodes are connected with the selected nodes on adjacent levels. All oriented trees of this kind form the feasible set of hierarchies (Riismaa 1993, 2003; Riismaa, Randvee and Vain 2003).

The illustration of this formalism is given in Fig.1 (Riismaa 2003).

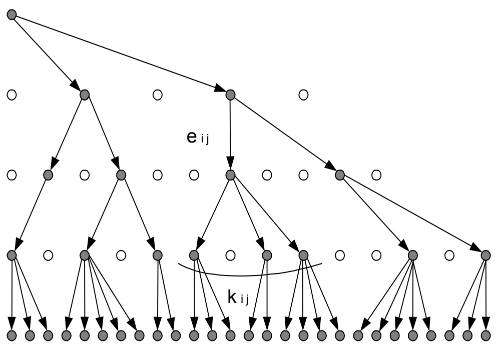

Figure 1 - Feasible set of structures

Suppose $m_i \times m_{i-1}$ matrix $Y_i = \left(y^i_{jr}\right)$ is an adjacent matrix of levels *i* and $i-1$ $(i = 1,...,s)$ where

$$y^i_{jr} = \begin{cases} 1, j - th\ element\ on\ level\ i\ connected \\ \ \ with\ r - th\ element\ on\ level\ i-1 \\ 0, otherwise \end{cases}.$$

Suppose m_0 is the number of 0-level elements (level of object).

Theorem 1. All hierarchies with adjacency matrixes of adjacent levels $\{Y_1,...,Y_s\}$ from the described set of hierarchies satisfy the condition

$$Y_s \cdot ... \cdot Y_1 = (\underbrace{1,...,1}_{m_0}) \qquad (1)$$

The assertion of this theorem is determined directly.

The general optimization problem is stated as a problem of selecting the feasible structure which corresponds to the minimum of total loss given in the separable-additive form:

$$\min \left\{ \sum_{i=1}^{s} \sum_{j=1}^{m_i} \mathrm{h}_{ij} \left(\sum_{r=1}^{m_{i-1}} d^i_{jr} y^i_{jr} \right) \middle| \ Y_s \cdot ... \cdot Y_1 = (\underbrace{1,...,1}_{m_0}) \right\}$$

over $Y_1,..., Y_s$.

(2)

Here $h_{ij}(\cdot)$ is an increasing loss function of *j*-th element on *i*-th level and d^i_{jr} is the element of $m_i \times m_{i-1}$ matrix D_i for the cost of connection between the *i*-th and (*i*-1)-th level.

The meaning of functions $h_{ij}(k)$ depends on the type of the particular system.

In this paper we suppose additionally that

$h_{ij}(0) = 0(i = 1,..,s; j = 1,...,m_i)$ and $h_{ij}(k)(i = 1,...,s; j = 1,...,m_i)$ are increasing functions.

By the optimization of the structure of multi-level tournament system, the loss inside the *j*-th tournament on *i*-th level is

$$h_{ij}(k_{ij}) = d^i_j k_{ij}(k_{ij} - 1),$$

where k_{ij} is the number of participants of *j*-th tournament of *i*-th league.

By complexity optimization of hierarchically connected subsystems, the loss inside the *j*-th set of partitioning on *i*-th level may be defined as follows:

$$h_{ij}(k) = \sum_{q=1}^{k} a_{ijq} \frac{k!}{q!(k-q)!}.$$

In this case, the value of the function $h_{ij}(k)$ describes the number of all nonempty subsystems inside the *j*-th set of partitioning on *i*-th level.

Mathematically, this problem is an integer programming problem with a non-continuous objective function and with a finite feasible set.

3. REDUCED PROBLEM OF OPTIMAL PARALLEL AND DISTRIBUTED PROCESSING SYSTEM

Here an important special case is considered, where the connection cost between the adjacent levels is the property of the supreme level: each row of the connection cost matrices between the adjacent levels consists of equal elements. It means that for each item on the next level the connection with all items on the previous level have the same costs:

$$d^i_{jr} = d^i_j \left(i = 1,...,s; j = 1,...,m_i; r = 1,...,m_{i-1}\right).$$

Now there is a possibility to change the variables and to represent the problem so that

$$d^i_{jr} = 1; \quad i = 1,...s; \; j = 1,...,m_i; \; r = 1,...,m_{i-1}.$$

Now the total loss depends only on sums $\sum_{r=1}^{m_{i-1}} y^i_{jr} = k_{ij}$, where k_{ij} is the number of edges beginning in the j-th node on i-th level.

In terms of tournament theory, the goal function doesn't depend how to distribute the winners on previous level between tournaments on the next level. But the goal function depends only how large are the tournaments. In terms of graph theory, the goal function doesn't depend what nodes connect but depends how many nodes to connect. Shortly, if additionally to change the variable, each connection between adjacent levels has the same cost.

Recognize also that $\sum_{j=1}^{m_i} k_{ij} = p_{i-1}, \; i = 1,...,s$, where p_i is the number of nodes on i-th level. If to suppose additionally that $h_{i1}(k) \le \cdots \le h_{im_i}(k)$ for each integer k, the general problem (2) transforms into the two mutually dependent phases:

$$\min\left\{\sum_{i=1}^{s} g_i\left(p_{i-1}, p_i\right) \,\middle|\, \left(p_1,..., p_{s-1},1\right) \in W^s\right\} \quad (3)$$

over $p_1,..., p_{s-1}$

where

$$g_i\left(p_{i-1}, p_i\right) = \min\left\{\sum_{j=1}^{p_i} h_{ij}\left(k_{ij}\right) \,\middle|\, \sum_{j=1}^{p_i} k_{ij} = p_{i-1}\right\}$$

over $k_{i1},...,k_{ip_i}$

$$W^s = \left\{\left(p_1,..., p_{s-1},1\right) \,\middle|\, 1 \le p_i \le p_{i-1}\right\}. \quad (4)$$

Free variables of the inner minimization (4) are used to describe the connections between the adjacent levels. Free variables of the outer minimization (3) are used for the representation of the number of elements at each level.

For solving problem (3), (4) double-cycle recursive optimization algorithms are constructed (Riismaa 2011). The inner cycle increases the number of elements inside of the current level by one unit, and outer cycle on each step increases the number of levels by one unit. On the each iteration, the one-parameter integer-programming problem must be solved.

4. ANALYTICAL METHOD OF SOLVING REDUCED PROBLEM FOR HORIZONTAL HOMOGENEOUS HIERARHIES

The hierarchy is called horizontal homogeneous if

$$h_{ij}\left(k\right) = h_i\left(k\right) \; \left(j = 1,...,m_i; i = 1,...,s\right). \quad (5)$$

Now (3), (4) transforms to

$$\min\left\{\sum_{i=1}^{s} g_i\left(p_{i-1}, p_i\right) \,\middle|\, \left(p_1,..., p_{s-1},1\right) \in W^s\right\} \quad (6)$$

over $p_1,..., p_{s-1}$

where

$$g_i\left(p_{i-1}, p_i\right) = \min\left\{\sum_{j=1}^{p_i} h_i\left(k_{ij}\right) \,\middle|\, \sum_{j=1}^{p_i} k_{ij} = p_{i-1}\right\}$$

over $k_{i1},...,k_{ip_i}$ (7)

$$W^s = \left\{\left(p_1,..., p_{s-1},1\right) \,\middle|\, 1 \le p_i \le p_{i-1}\right\}.$$

This statement has some advantages from the point of view of the optimization technique. It is possible to adapt effective methods of the convex programming for solving outlined special cases.

The function $f : X \to R$, $X \subset R^n$, is called discrete-convex (Riismaa 1993; Murota 2003) if for all $x_i \in X (i = 1,..., n+1); \lambda_i \ge 0 (i = 1,..., n+1)$ and $\sum_{i=1}^{n+1} \lambda_i = 1; \sum_{i=1}^{n+1} \lambda_i x_i \in X$ holds

$$f\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) \le \sum_{i=1}^{n+1} \lambda_i f(x_i).$$

Suppose additionally, that $h_i(k)\,(i = 1,...,s)$ in (7) are discrete-convex functions, and $h_i\left(0\right) = 0\left(i = 1,...,s\right)$.

Then $\sum_{i=1}^{s} g_i\left(p_{i-1}, p_i\right)$ in (6) is a discrete-convex function (Riismaa 2011).

Now it is possible to solve the inner minimization problem (3) (to find the optimal connections between the adjacent levels) analytically:

$$g_i(p_{i-1}, p_i) =$$
$$= \left(p_i \cdot \left(\left[\frac{p_{i-1}}{p_i} \right] + 1 \right) - p_{i-1} \right) \cdot h_i\left(\left[\frac{p_{i-1}}{p_i} \right] \right) +$$
$$+ \left(p_{i-1} - p_i \cdot \left[\frac{p_{i-1}}{p_i} \right] \right) \cdot h_i\left(\left[\frac{p_{i-1}}{p_i} \right] + 1 \right) \qquad (8)$$
$$(i = 1, \ldots, s-1),$$
$$g_s(p_{s-1}, 1) = h_s(p_{s-1}), [p] \text{ - integer part of } p.$$

Denote $p_i \cdot \left(\left[\frac{p_{i-1}}{p_i} \right] + 1 \right) - p_{i-1} = p_i^*$ and

$$p_{i-1} - p_i \cdot \left[\frac{p_{i-1}}{p_i} \right] = p_i^{**}.$$

Recall

$$k_{ij} = \left[\frac{p_{i-1}}{p_i} \right] \ (\mathrm{j} = 1, \ldots, \mathrm{p}_\mathrm{i}^*),$$
$$k_{ij} = \left[\frac{p_{i-1}}{p_i} \right] + 1 \ \left(j = p_i^* + 1, \ldots, p_i^* + p_i^{**} \right)$$
$$(i = 1, \ldots, s)$$

Certainly $p_i^* + p_i^{**} = p_i$.

To complete the solving of problem (3), (4) it is enough to use (8) for outer optimization problem (3):

$$\min \left\{ \sum_{i=1}^{s} \left(\begin{array}{l} \left(p_i \cdot \left(\left[\frac{p_{i-1}}{p_i} \right] + 1 \right) - p_{i-1} \right) \cdot h_i\left(\left[\frac{p_{i-1}}{p_i} \right] \right) + \\ \left(p_{i-1} - p_i \cdot \left[\frac{p_{i-1}}{p_i} \right] \right) \cdot h_i\left(\left[\frac{p_{i-1}}{p_i} \right] + 1 \right) \end{array} \right) \middle| \ (p_1, \ldots, p_{s-1}, 1) \in W^s \right\}$$
$$\text{over } p_1, \ldots, p_{s-1} \qquad .(9)$$

This problem can be solved with method of recursive optimization (Riismaa 2011).
If $h_i(\cdot)(i = 1, \ldots, s)$ are discrete-convex functions, the problem (9) is a discrete-convex programming problem and can be solved with method of recursive optimization or with method of local searching (Riismaa 2003).
Is possible to approximate the functions (8) with

$$g_i(z_{i-1}, z_i) = z_i h_i\left(\frac{z_{i-1}}{z_i} \right) \ (i = 1, \ldots, s), z_s = 1.$$

Here $z_i \ (i = 1, \ldots, s)$ are not integer and

$k_{ij} = k_i = \frac{z_{i-1}}{z_i} (i = 1, \ldots, s)$ are not integer.

With this approximation and with (5) the problem (3) – (4) transforms to

$$\min_{z_1, \ldots, z_{s-1}} \left\{ \sum_{i=1}^{s} z_i h_i\left(\frac{z_{i-1}}{z_i} \right) \middle| \begin{array}{l} z_0 \geq z_1 \geq \ldots \geq z_{s-1} \geq 1; \\ z_i \in R^+ (i = 1, \ldots, s-1) \end{array} \right\}$$
(10)

If $h_i(\cdot)(i = 1, \ldots, s)$ are convex functions, then problem (10) is convex programming problem.
Unfortunately there is difficult to estimate this approximation error.

5. ILLUSTRATIVE EXAMPLE: OPTIMIZATION THE STRUCTURE OF MULTI-LEVEL PARALLEL AND DISTRIBUTED PROCESSING SYSTEM

Consider the processing of *n* parts (Riismaa 2011). In case of one processing unit, the overall processing and waiting time for all n parts is proportional to n^2 and is a quickly increasing function. For this reason, the hierarchical system of processing can be suitable. From zero-level (level of object) the parts will be distributed between p_1 first-level processing units and processed (aggregated, packed etc.) by these units. After that, the parts will be distributed between p_2 second-level processing units and processed further and so on. From p_{s-1} (*s* - 1)-level, the units will be sent to the unique s-level unit and processed finally. The cost of processing and waiting on level i is approximately

$$g_i(p_{i-1}, p_i) = (d_i \, l_{i-1} p_{i-1} / p_i)^2 \, p_i + a_i p_i$$
$$(i = 1, \ldots, s).$$

Here l_i is the number of aggregates produced by one robot on level i (a number of boxes for packing unit), d_i is a loss unit inside level i, and a_i is the cost of *i*-th level processing unit. The variable parameters are the number of processing units on each level $p_i(i = 1, \ldots, s)$.

The goal is to minimize the total loss (processing time, waiting time, the cost of processing units) over all levels:

$$\min \sum_{i=1}^{s} ((d_i l_{i-1})^2 \left(\begin{array}{l} \left(p_i \left(\left[\frac{p_{i-1}}{p_i} \right] + 1 \right) - p_{i-1} \right) \left[\frac{p_{i-1}}{p_i} \right]^2 + \\ + \left(p_{i-1} - p_i \left[\frac{p_{i-1}}{p_i} \right] \right) \left(\left[\frac{p_{i-1}}{p_i} \right] + 1 \right)^2 \end{array} \right) + a_i p_i)$$

over natural $p_i (i = 1,..., s)$. Here $[p]$ is the integer part of p .

6. CONCLUSION

Many discrete or finite hierarchical structuring problems can be formulated mathematically as a multi-level partitioning procedure of a finite set of nonempty subsets. This partitioning procedure is considered as a hierarchy, nodes of hierarchy correspond to the subsets of partitioning and the relation of containing of subsets defines the arcs of the hierarchy. The feasible set of structures is a set of hierarchies (oriented trees) corresponding to the full set of multi-level partitioning of given finite set.

In mathematical modeling, the choice of variables is important problem. For some special cases here two types of variables are defined. The variables of inner minimization are used to describe the connections between the adjacent levels. The variables of outer minimization are used for the presentation of number of elements on each level. The formalism considered here, enables to state the structure optimization problem as a two-phase mutually dependent discrete optimization problem and to construct some classes of effective solution methods.

Examples of problems of this class are aggregation problems, structuring of decision-making systems, database structuring, multi-level tournament systems, multi-level distribution systems.

The approach is illustrated by a multi-level production system example.

REFERENCES

Laslier, J.- F., 1997. *Tournament solutions and majority voting*. Springer, Berlin: Springer.

Mesarovic, M. D., Macko, D. and Takahara, Y., 1970. Theory of hierarchical multi-level systems. New York: Academic Press.

Murota, K., 2003. *Discrete convex analysis*. Philadelphia: SIAM.

Riismaa, T., 1993. Description and Optimization of Hierarchical Systems. *Automation and Remote Control*, 12, 146 - 152

Riismaa, T., 2003. Optimization of the Structure of Fuzzy Multi-Level Decision-Making System. *Proceedings of International Conference on Modelling and Simulation of Business Systems*, pp. 31 – 34. May 13 – 14, 2003 Vilnius, (Lithuania).

Riismaa, T., Randvee, I. and Vain, J. 2003. Optimization of the Structure of Multi-Level Parallel Assembling. *Proceedings of 7th IFAC Workshop on Intelligent Manufacturing Systems*, pp. 235 – 238. April 6 –8, Budapest, (Hungary).

Riismaa, T., 2011. Recursive algorithm for optimization the structure of complicated logistic, manufacturing and processing environments. *Proceedings of 13rd International Conference on Harbour, Maritime & Multimodal Logistic Modeling nd Simulation*, pp. 99 – 104. September 12 – 14, Rome (Italy).

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