# Designing irregular gears via splines 

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#### Abstract

To construct a new machine, to improve the existing one or to reconstruct the worn up, damaged machine/tool, we need its theoretical model, in both functional and material issues, and the means to produce its final form. Contemporarily, there are used computers in all the stages of the production process, as well as in so-called inverse tasks, when we want to reconstruct (the parts of) machines. It demands still more and more precise mathematical description. In this paper we discuss such description via spline curves (composed of third degree polynomial arcs represented parametrically) and we apply it to gears/pulleys of nontypical irregular profiles.


Keywords: spline, noncircular wheel, reverse engineering, computer aided design

## 1. MOTIVATION AND INTRODUCTION

The usage of noncircular gears and pulleys makes possible to have better characteristics of the transmission, e.g., the changeable kinematical features in the gear ratio and the velocity. The required degree of speed variability is obtained by the use of pulleys constructed with wheel rims having shapes of ellipses, ovals or other non-circular disks. There is already well recognized the usage of chains in such drives. On the other hand, the usage of toothed belts in such transmission systems is not described in details. That is due to the different kinematics and coupling characteristics of toothed belts and pulleys as compared with chain drives. In a chain drive the driven strand can be slack, whereas in case of a toothed belt drive it must be tight. It makes that there must be met two conditions. The first condition concerns the length of the belt: it must be equal to the length of the envelope. In order to ensure the correct operation of a variable-speed transmission system, the active and passive sections of the belt must be tightened by an appropriate constant force. The second condition of the correct work of a variable-speed transmission system is to ensure the cyclycity of its movement. This cyclicity is absolutely required during the circular motion of machine elements. The circumference of every wheel has to be an integer multiple of the pitch (as always, a metric pitch is the distance between neighboring belt teeth
based on millimeters). Thus, one is able to determine the average transmission ratio of the system as the relation of circumferences of wheels or the number of their teeth. The cyclicity of drives can be guaranteed only by toothed belts having following property: during the operation the plastic strain varies slightly. At the same time the belts must be initially pre-tightened in order to avoid the slip of the belt as well as the skip on the teeth of the wheels.

If both conditions, a constant tension of the belt and the cyclicity condition, are met simultaneously, one is able to search for a design of a variable-speed transmission system as that shown in Fig.1.


Fig. 1. Belt and pulley transmission systems installed in LEDM (Laboratory for Experimental Design and Manufacturing at Faculty of Machines and Transport, Poznań University of Technology); the system has two circular wheels, one noncircular wheel and one eccentrically mounted wheel

The problem at hand is widely studied in the case of regular noncircularity, e.g., when there are applied elliptical or trochoidal wheels, both types being welldescribed mathematically. In this study we deal with irregular gears in case when their profiles are not covered by well-known mathematical equations. The design of such gears involves more advanced
techniques in both technological and mathematical aspects, in the last case it is often needed a numerical treatment.

The design of noncircular gears and pulleys working in tooth transmissions is widely described in literature, see, e.g., Laczik (2003), Danieli et al. (2005), MingFeng Tsay and Zhang-Hua Fong (2005), Bair et al. (2007), JianGang Li et al. (2007), Bair (2009). In most cases there are considered regular noncircular wheels, i.e., the gears of the elliptical and cycloidal profile; only such plane curves are discussed in the book by Litvin and Fuentes (2004), a bible for gear designers. In last years there are undertaken practical experiments and theoretical considerations concerning non-typical irregularly shaped elements of belt/chain drivers. As far as we know, the literature dedicated to such non-typical toothing is rather modest, it is treated in, e.g., Li Xin Cao et al. (2002), Krawiec (2005). It is also discussed by Gajda, Krawiec and Marlewski (2008), where Bézier curves are applied to describe such profiles.

A mathematical description of the profile is necessary when there are used modern machines applying CNC (computer numerical control). These machines manufacture, for instance by laser devices or compressed water streams, elements of machines and they cut a desired profile via moving their cutting tools along the trajectory which has to be defined mathematically.

## 2. SPLINE CURVE INTERPOLATING <br> A PROFILE TO BE RECOVERED

In more details, we present the way at which we obtained the spline description of the closed curve passing through given $m$ points $P_{j}=\left(x_{j}, y_{j}\right),(j=1,2, \ldots$, $m$ ), sitting on the profile of a non-typical gear, as that seen in Fig. 2 below.


Fig.2. Belt pulley manufactured in LEDM
The coordinates of thousands of such points are provided, as pairs of two numbers, $x_{j}$ and $y_{j}$, by a CMM (coordinate measuring machine, see Fig.3) and a designer/constructor decides which ones of them are taken in aim to get a model; in the case reported here there were taken 24 extreme points of the outside


Fig.3. Measuring geometrical parameters of a non-circular pulley on the CMM

Contura G2 from Carl Zeiss


Fig.4. A sample profile and points $P_{j}$ ( $j=1,2, \ldots, m ; m=24$ ), sitting on its extreme envelope
envelope of the profile; you can see these points in Fig. 4 (and we refer to them as measured points). These coordinates form two one-column vectors, namely

$$
X=\left[x_{1}, x_{2}, \ldots, x_{m}\right]^{\mathrm{T}}, Y=\left[y_{1}, y_{2}, \ldots, y_{m}\right]^{\mathrm{T}}
$$

$X$ and $Y$ store the abscissas and the ordinates, respectively.

Since the profile is closed, it is natural to augment both vectors $X$ and $Y$ by the element equal to $x_{1}$ and $y_{1}$, respectively. This way we have so-called abscissavector $X$ and ordinate-vector $Y$,

$$
X=\left[x_{1}, x_{2}, \ldots, x_{m}, x_{m+1}\right]^{\mathrm{T}}, Y=\left[y_{1}, y_{2}, \ldots, y_{m}, y_{m+1}\right]^{\mathrm{T}},
$$

where $x_{m+1}=x_{1}$ and $y_{m+1}=y_{m}$. Obviously, we can say that there are given no $m$ points $P_{j}$, but there are given $m+1$ points

$$
P_{j}=\left(x_{j}, y_{j}\right),(j=1,2, \ldots, m, m+1)
$$

where $P_{m+1}=P_{1}$.
The augmenting we did simplifies the presentation of the method below. This method provides the equation of the curve interpolating the sequence ( $P_{1}, P_{2}, \ldots, P_{m}, P_{m+1}$ ) of measured points.

Obviously, the profile at hand, as well as the sequence ( $P_{1}, P_{2}, \ldots, P_{m}, P_{m+1}$ ) of measured points, can not be entirely covered by the equation of the form $y=f(x)$. Fortunately, it can be describe parametrically; we can find the functions $x=x(t)$ and $y=y(t)$, both in the variable $t$, such that $x=x(t)$ interpolates the abscissa-vector $X$ and $y=y(t)$ interpolates the ordinatevector $Y$. Then the entire curve is governed by the equation

$$
s(t)=[x(t), y(t)],
$$

where $t$ runs an appropriate interval and $s\left(t_{j}\right)=P_{j}$ for appropriately chosen values $t_{j}$ of the variable $t$. These values, $t_{j}$, are called knots, or nodes, of interpolation. Below it will appear clear that it is very convenient to deal with the knots $t_{j}=j$. These knots determine the intervals $\left\langle t_{j}, t_{j+1}\right\rangle=\langle j, j+1\rangle, j=1,2, \ldots, m$; every one of them is called an elementary, or basic, interval (for the method we apply below).

Since we want to have an interpolating spline curve, we look for splines $x=x(t)$ and $y=y(t)$ satisfying collocation conditions

$$
x\left(t_{j}\right)=x_{j} \text { and } y\left(t_{j}\right)=y_{j} \text { for } j=1,2, \ldots, m
$$

As in numerous applications, we will find the third degree spline interpolation. So we look for polynomials of third degree, $f_{j}$ and $g_{j}(j=1,2, \ldots, m)$, to interpolate the abscissa-vector $X$ and ordinate-vector $Y$, respectively. For instance, on the $j$-th basic interval $<t_{j}, t_{j+1}>$ the spline $x=x(t)$ can be taken in the form

$$
x(t)=a_{j}+b_{j} \cdot\left(t-t_{j}\right)+c_{j} \cdot\left(t-t_{j}\right)^{2}+d_{j} \cdot\left(t-t_{j}\right)^{3}
$$

and it is clear that it is defined by its coefficients, $a_{j}, b_{j}$, $c_{j}, d_{j}$.

Analogously, the $j$-th part of the spline $y=y(t)$ is determined by other quadruple, $\alpha_{j}, \beta_{j}, \gamma_{j}, \delta_{j}$, forming the coefficients of the linear combination of the same basic functions as above, i.e.,

$$
t \rightarrow\left(t-t_{j}\right)^{k-1}, k=1,2,3,4 .
$$

The set of these four functions can be called a Herriot-
Reinch basis. This basis is a key point to find all desired coefficients in HeRA, a Herriot-Reinsch algorithm (see Herriot and Reinsch 1973 and, e.g., Krawiec and Marlewski 2011).

In this algorithm we first calculate multipliers of the second power, $\left(t-t_{j}\right)^{2}$, next we use some combinations of them to get the multipliers of $\left(t-t_{j}\right)^{3}$ and $\left(t-t_{j}\right)^{1}$. At last, by the collocation condition the free terms are produced at once: it is $a_{j}=x_{j}$ for the spline $x=x(t)$ and, as easily as here, $\alpha_{j}=y_{j}$ for the spline $y=y(t)$.

The multipliers $c_{j}$ of $\left(t-t_{j}\right)^{2}$ standing in the spline $x=x(t)$ are solutions of the system of linear algebraic equations

$$
A \cdot C=\xi,
$$

where $A$ is the matrix of the system at hand, $c$ comprises the coefficients to be calculated, $\xi$ is the vector of right sides, so

$$
\begin{aligned}
c & =\left[c_{1}, c_{2}, \ldots, c_{m}\right]^{\mathrm{T}}, \\
\xi & =\left[\xi_{1}, \xi_{2}, \ldots, \xi_{m}\right]^{\mathrm{T}},
\end{aligned}
$$

$$
A=\frac{1}{3} \cdot\left[\begin{array}{cccccccc}
4 & 1 & 0 & 0 & \ldots & 0 & 0 & 1 \\
1 & 4 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 4 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & 4 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & .0 & \ldots & 4 & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 1 & 4 & 1 \\
1 & 0 & 0 & 0 & \ldots & 0 & 1 & 4
\end{array}\right],
$$

The matrix $A$ of order $m$ and the vector $\xi$ are built according to the definition of the spline with the additional requirement saying that this spline has to be periodic. This periodicity makes that the respective spline curve is closed and, at the same time, it puts both 1 's in the left down and right upper corners (and there is the only difference with respect to the natural spline interpolation, where instead of these 1's we have 0 's).

The elements of the right-side vector $\xi$ are

$$
\begin{aligned}
& \xi_{1}=\rho_{1}-\rho_{m}, \\
& \xi j=\rho_{j}-\rho_{j-1} \quad \text { for } j=2,3, \ldots, m,
\end{aligned}
$$

where $\rho_{j}=x_{j+1}-x_{j} \quad$ for $j=1,2, \ldots, m$.
Solution $c$ calculated, we get the vectors $d=\left[d_{j}\right]$ and $b=\left[b_{j}\right]$ by the formulas

$$
\begin{aligned}
& d j=\gamma_{j+1}-\gamma_{j} \quad \text { for } j=1,2, \ldots, m-1, \\
& d_{m}=\gamma_{1}-\gamma_{m}, \\
& b j=r_{j}-2 \gamma_{j}-\gamma_{j+1} \text { for } j=1,2, \ldots, m-1, \\
& d_{m}=r_{m}-2 \gamma_{m}-\gamma_{1},
\end{aligned}
$$

where $\quad \gamma_{j}:=c_{j} / 3$.
If necessary, go to Krinze $\beta$ (2006) to see how the system $A \cdot C=\xi$ is built, and to Carnahan et al. (1969) to state that $A$ is symmetric, irreducible and positively determined. The last fact was proved by de Boor and DeVore (1985) for arbitrarily spaced knots, but in our case we have a regular mesh and the determinants for $m=3,4,5,6,7, . ., 24,25$ are equal $18,64,242,900$, $3362,12544, \ldots, 17767236614400,66308229755042$ (it seems that there is still unknown a general formula for the value of $m$-th determinant).

In view of above remarks, the system $A \cdot c=\xi$ can be solved iteratively by Jacobi method. It can be also solved directly, $c=A^{-1} \cdot \xi$, by SOSes (symbolically oriented systems, as Derive from Texas Instruments, Inc., Mathematica from Wolfram Research, Inc.) without any roundings. Let's give an example: with $m=5$ there is

$$
A^{-1}=\frac{1}{8} \cdot\left[\begin{array}{rrrr}
7 & -2 & 1 & -2 \\
-2 & 7 & -2 & 1 \\
1 & -2 & 7 & -2 \\
-2 & 1 & -2 & 7
\end{array}\right]
$$

Fortunately, the system $A \cdot c=\xi$ can be also solved by direct methods (including the basic one: Gauss elimination method) in NOSes (numerically oriented systems, e.g. Pascal, C++), because its matrix
$A=\left[a_{j, k}\right]_{j, k=1,2, \ldots, m}$ is well-conditioned; for instance, its condition number generated by maximal norm, $\|A\|=\max \left\{\left|a_{j, k}\right|: j, k=1,2, \ldots, m\right\}$, does not exceed 1.16 when $m \leq 51$.

The construction of the equation $y=y(t)$ concerning the ordinate-vector $Y$ is identical as that presented above, and this way it is reduced to solve the linear system $A \cdot \gamma=\eta$ with the same matrix $A$. Therefore we can treat these two tasks at once, namely instead of solving two systems of linear algebraic equations we solve the system

$$
A \cdot W=F,
$$

where both $W=[c \mid \gamma], F=[\xi \mid \eta]$ are $m \times 2$-matrices, their columns are vectors $c$ and $\gamma$, and the vectors $\xi$ and $\eta$ are determined by the abscissa-vector $X$ and the ordinatevector $Y$, respectively,
$A$ is as above, so it is worthy to get the inverse matrix $A^{-1}$ or to apply any other method simultaneously to the pairs ( $c, \xi$ ) and $(\gamma, \eta)$ composed of columns involved in matrices $W$ and $F$.
The solution $W$ of this system, via its columns $c$ and $\gamma$, yields the other coefficients ( $b_{j}, d_{j}$ etc.) of desired functions

$$
x=x(t), y=y(t)
$$

covering the positioning of abscissas and ordinates related to the equidistantly distributed knots $t_{j}=j$ related to the variable $t, j=1,2, \ldots, m+1$. This way we finally get the desired spline

$$
s(t)=[x(t), y(t)] .
$$

where $t$ is the parameter; for $t \in\langle j, j+1\rangle$ we have the $j$-th fragment of the final spline curve, $j=1,2, \ldots, m$,
$x=x(t)$ is the function in the variable $t$; it covers the behavior of the spline at hand along the horizontal axis $O x$,
$y=y(t)$ is the function in $t$; it describes the changes along the vertical axis $O y$.

## 3. ANGLES AT WHICH A TOOL CUTS A PLATE

Obviously, to cut a desired pulley, or gear, off a steel (an aluminium alloy etc.) plate, the cutting tool controlled by a computer has to have not only the equation of the outside profile, but also the depth of toothing and the direction at which the a laser (a water jet etc.) moves. The depth, in millimeters, is the same for every tooth and it is simply passed in through the control panel. The directions are defined by angles, in grades, at which the tool is oriented with respect to the zero-direction (it is set just when the cutting machine starts its work, see Fig.5). The angles vary from tooth to tooth and they are passed to the cutting machine controller through a specialized programme as the sequence of numbers accompanying the coordinates of nodes $P_{j}$.


Fig.5. Forming the non-regular noncircular belt pulley on CNC milling machine Deckel Maho from DMG

$$
\begin{aligned}
& \text { Let } \\
& \qquad \begin{array}{l}
x(t)=a_{j}+b_{j} \cdot(t-j)+c_{j} \cdot(t-j)^{2}+d_{j} \cdot(t-j)^{3}, \\
y(t)=\alpha_{j}+\beta_{j} \cdot(t-j)+\gamma_{j} \cdot(t-j)^{2}+\delta_{j} \cdot(t-j)^{3},
\end{array}
\end{aligned}
$$

be the parametric equation of a $j$-th part of the spline curve $[x, y]=s_{j}(t)$ obtained above; it says that this equation covers this curve when $j$ runs from the point $P_{j}$ to the point $P_{j+1}$. Since for every $t$ where the derivative $x^{\prime}(t)$ does not vanish there holds true

$$
\frac{d y}{d x}=\frac{\frac{d y}{d t}}{\frac{d x}{d t}}=\frac{\beta_{j}+2 \gamma_{j} \cdot(t-j)+3 \delta_{j} \cdot(t-j)^{2}}{b_{j}+2 c_{j} \cdot(t-j)+3 d_{j} \cdot(t-j)^{2}},
$$

so at $P_{j}$ the tangent line has the slope $\beta_{j} / b_{j}$ and, in consequence,

$$
\varphi_{j}=\operatorname{arccot} \frac{-b_{j}}{\beta_{j}}
$$

is the angle under which, at the point $P_{j}$, the normal line to the curve $s=s_{j}(t)$ is inclined to the zero-direction of the cutting machine.

Obviously, the same straight line is perpendicular to the adjacent spline fragment $s=s_{j-1}(t)$ (for $j=1$ we identify $s_{0}=s_{m+1}$ ). If $b_{j}=0$, then the tangent line is parallel do the axis $O x$ and $\varphi_{j}$ is $90^{\circ}$ or $270^{\circ}$ depending on the sign of the coefficient $\beta_{j}$.

In Fig. 6 there are traced segments of perpendicular lines to the spline curve. For instructive purpose, a segment of the normal at the knot $P_{2}$ is traced longer than other segments. Since here the zero-direction of the cutting machine coincides with the direction of the Oy axis and the direction at which the object in forming is clockwise, we have $\varphi_{2}=170.489^{\circ}$ (the normal at $P_{2}$ is inclined to the axis Oy upon the angle $170^{\circ} 29^{\prime}$ ) and, e.g. $\varphi_{1}=4.8545^{\circ}=4^{\circ} 51^{\prime}, \varphi_{24}=22.8724^{\circ}=22^{\circ} 52^{\prime}, \varphi_{20}$ $=86.9719^{\circ}=86^{\circ} 58^{\prime}$.


Fig.6. Nodes $P_{j}$ (marked by values of their index $j: 1,2$, $3, \ldots, 24$ ) of the interpolatory spline, the spline curve itself and the normal lines to it passing through the nodes (figure produced in Derive 5 for Windows)

## 5. MANUFACTURING IRREGULAR PULLEYS AND GEARS

Belt pulleys, as well as gears, are manufactured by the profiling (a.k.a. shaping) method or by the envelope method; they both are classified as matching techniques. The disadvantage of the profiling technique is that it needs to use few cutting tools and, practically always, an additional mechanical polishing has to be performed. Moreover, manufactured gears have so-called pitch error, which results from nature of process. By contrast, the envelope technique, a.k.a. the direct generation of noncircular gears, requires the design of non-typical manufacturing process and application of numerical controlled slotting machine; as several years earlier, the last one treatment is still not very common, see Kujawski (1992).

Noncircular belt pulleys can be also shaped by abbrasive water jets, a.k.a. watersaws, which are tools capable of slicing into metal, or other material, using a jet of water (usually enriched with an abrasive substance) at high velocity and pressure.

Another way to manufacture gears and pulleys is the laser cutting (in particular by lasers where $\mathrm{CO}_{2}$ is the lasing material), but here only relatively thin gears can be obtained, see, e.g., Krawiec (2009).

As we read in Krawiec (2010), a good alternative method to all techniques mentioned above is the application of universal CNC milling machine with set of end mills, as well as rapid prototyping and rapid manufacturing methods (e.g., 3D-printing, FDM, SLS) to get gear wheels. A relatively new idea of gears forming is the usage of numerical controlled milling machine, where there is installed special two cutting edges tool. In accordance with this idea the manufacturing process of gear wheel is composed of the
following movements: tool rotation in relation to spindle axis, rotation and displacement of numerical controlled table, where initially shaped blank is positioned. In the referred method the basic task in the elaboration of the controlling program is the proper correlation between the movement of the table and the rotation of the cutting head. This relation can be doubtlessly stated by mathematical formulas and the spline description presented in this paper provides it.

## 4. FINAL COMMENTS

Taking into account the regimes obligatory for the manufacturing belt pulleys and gears we derived the equation of spline curve passing through given points chosen, by a designer, from points provided by any scanning machine. The accuracy at which these data are gathered is fairly better than the ranges at which the desired gear/pulley has to be manufactured; the respective errors are even 0.001 mm in CMM and 0.1 mm in cutting process if the diameter ranges between 50 and 100 mm . That's why we did not smooth the scanned data and we did not smooth the spline determined on given points. The second procedure, aiming in the smoothing of a spline generated by given sequence of points $P_{j}$, is discussed, a.o., in Reinsch (1967) and, moreover, Hutchinson and de Hoog (1985).

Since invented, the spline interpolation is widely applied not only in civil engineering and mechanics, but also in such areas as statistics, geometry and (rail)way planning - see Kamenschykova (2008), Kranjc (2009) and Moreb (2009). When applied to the problem considered in this paper, it also provides a fast way to determine the angles of the cut and that's why it is more friendly to the practice than Bézier approximation (although the formulas involved in the spline description are more complex that the simply, elegant dependency taking place in Bézier approach).

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