POINTWISE COMPLEATNESES AND POINTWISE DEGENERACY OF STANDARD AND POSITIVE HYBRID LINEAR SYSTEMS DESCRIBED BY THE GENERAL MODEL

Tadeusz Kaczorek

Faculty of Electrical Engineering
Bialystok University of Technology
Bialystok, Poland

kaczorek@isep.pw.edu.pl

ABSTRACT

Necessary and sufficient conditions for the pointwise completeness and pointwise degeneracy of the standard and positive hybrid linear systems described by the general model are established. It is shown that the standard general model is always pointwise complete and it is not pointwise degenerated and the positive general model is pointwise complete if and only if its matrix A_2 is diagonal.

Keywords: pointwise completeness, degeneracy, standard, positive, general model, hybrid system

1. INTRODUCTION

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs (Farina and Rinaldi 2000; Kaczorek 2002).

The most popular models of two-dimensional (2D) linear systems are the discrete models introduced by Roesser (1975), Fornasini and Marchesini (1976, 1978), and Kurek (1985). The models have been extended for positive systems. An overview of positive 2D system theory has been given in the monograph (Kaczorek 2002).

A dynamical system described by homogenous equation is called pointwise complete if every given final state of the system can be reached by suitable choice of its initial state. A system which is not pointwise complete, is called pointwise degenerated.

The pointwise completeness and pointwise degeneracy belong to the basic concepts of the modern control theory of 2D linear systems and they play important role specially in positive 2D linear systems.

The pointwise completeness and pointwise degeneracy of linear continuous-time system with delays have been investigated in (Olbrot 1972; Popov 1972), of discrete-time and continuous-time systems of fractional order in (Busłowicz 2008; Kaczorek and Busłowicz 2009) and of positive discrete-time systems with delays in (Busłowicz, Kociszewski and Trzasko 2006; Choundhury 1972). The pointwise completeness of linear discrete-time cone-systems with delays has been analyzed in (Popov 1972). The pointwise completeness and pointwise degeneracy of standard and positive linear systems with state-feedbacks have been investigated in (Kaczorek 2010a; Kaczorek 2009).

The pointwise completeness and pointwise degeneracy of 2D standard and positive Fornasini-Marchesini models have been addressed in (Kaczorek 2010b).

Positive 2D hybrid linear systems have been introduced in (Kaczorek 1998; Kaczorek 2007; Kaczorek 2002) and positive fractional 2D hybrid linear systems in (Kaczorek 2007). Comparison of different method of solution to 2D linear hybrid systems has been given in (Kaczorek, Marchenko and Sajewski 2008). Realization problem for positive 2D hybrid systems has been addressed in (Kaczorek 2008b).

In this paper the pointwise completeness and pointwise degeneracy of standard and positive hybrid linear systems described by the general model will be addressed.

The structure of the paper is the following. In section 2 the pointwise completeness and the pointwise degeneracy of the standard general model is investigated. Necessary and sufficient conditions for the positivity and the pointwise completeness, pointwise degeneracy of the general model are established in section 3. Concluding remarks are given in section 4.

In the paper the following notation will be used. The set of $n \times m$ real matrices will be denoted by $\Re^{n \times m}$ and $\Re^n = \Re^{n \times 1}$. The set of $n \times m$ real matrices with nonnegative entries will be denoted by $\Re_+^{n \times m}$ and $\Re_+^n = \Re_+^{n \times 1}$. The $n \times n$ identity matrix will be denoted by I_n and the transpose will be denoted by T.

2. POINTWISE COMPLETENESS AND POINTWISE DEGENERACY OF STANDARD GENERAL MODEL

Consider the autonomous general model

$$\dot{x}(t,i+1) = A_0 x(t,i) + A_1 \dot{x}(t,i) + A_2 x(t,i+1)$$
$$t \in \Re_+ = [0,+\infty], \quad i \in Z_+ = \{0,1,...\} \quad (2.1)$$

where $\dot{x}(t,i) = \dfrac{\partial x(t,i)}{\partial t}$, $x(t,i) \in \Re^n$, $u(t,i) \in \Re^m$,

$y(t,i) \in \Re^p$ are the state, input and output vectors.

Boundary conditions for (2.1) are given by

$$x_1(0,i) = x_i, \; i \in Z_+ \text{ and } x(t,0) = x_{t0}, \dot{x}(t,0) = x_{t1}, \; t \in \Re_+ \, (2.2)$$

Definition 2.1. The general model (2.1) is called pointwise complete at the point (t_f, q) if for every final state $x_f \in \Re^n$ there exist boundary conditions (2.2) such that $x(t_f, q) = x_f$.

Theorem 2.1. The general model (2.1) is always pointwise complete at the point (t_f, q) for any $t_f > 0$ and $q = 1$.

Proof. From (2.1) for $i = 0$ we have

$$\dot{x}(t,1) = A_2 x(t,1) + F(t,0) \quad (2.3)$$

where

$$F(t,0) = A_0 x(t,0) + A_1 \dot{x}(t,0) = A_0 x_{t0} + A_1 x_{t1}. \quad (2.4)$$

Assuming $x_{t0} = 0$, $x_{t1} = 0$ we obtain $F(t,0) = 0$ and from (2.3)

$$x(t,1) = e^{A_2 t} x(0,1). \quad (2.5)$$

Substituting $t = t_f$ and $q = 1$ we obtain

$$x_f = e^{A_2 t_f} x(0,1) \quad (2.6)$$

and

$$x(0,1) = e^{-A_2 t_f} x_f. \quad (2.7)$$

Therefore, for any final state x_f there exist boundary conditions $x_{t0} = 0$, $x_{t1} = 0$ and $x_1 = e^{-A_2 t_f} x_f$ such that $x(t_f,1) = x_f$ since the matrix $e^{-A_2 t_f}$ exists for any matrix A_2 and any $t_f > 0$. □

From theorem 2.1 we have the following corollaries.

Corollary 2.1. Any general model (2.1) is pointwise complete at the point $(t_f, 1)$ for arbitrary $t_f > 0$.

Corollary 2.2. The pointwise completeness of the general model at the point $(t_f, 1)$ is independent of the matrices A_0 and A_1 of the model.

Definition 2.2. The general model (2.1) is called pointwise degenerated at the point (t_f, q) in the direction

v if there exist a nonzero vector $v \in \Re^n$ such that for all boundary conditions (2.2) the solution of the model for $t = t_f$, $i = q$ satisfies the condition $v^T x(t_f, q) = 0$.

Theorem 2.2. The general model (2.1) is not pointwise degenerated at the point $(t_f, 1)$ for any $t_f > 0$.

Proof. Using the solution of (2.3)

$$x(t,1) = e^{A_2 t} x(0,1) + \int_0^t e^{A_2(t-\tau)} F(\tau,0) d\tau \quad (2.8a)$$

we obtain

$$v^T x(t,1) = v^T e^{A_2 t} x(0,1) + \int_0^t v^T e^{A_2(t-\tau)} F(\tau,0) d\tau \quad (2.8b)$$

where $F(t,0)$ is defined by (2.4). From (2.8b) it follows that does not exist a non-zero vector $v \in \Re^n$ such that for all boundary conditions (2.2) $v^T x(t_f,1) = 0$ since the matrix $e^{A_2 t_f}$ is nonsingular for every matrix A_2 and $t_f > 0$. □

Example 2.1. Consider the general model (2.1) with the matrices

$$A_0 = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}. \quad (2.9)$$

Find the boundary conditions (2.2) at the point $(t_f, q) = (1,1)$ for $x_f = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Taking into account that the eigenvalues of A_2 are $\lambda_1 = -1$, $\lambda_2 = -2$ and using the Sylvester formula we obtain

$$e^{-A_2 t} = \frac{A_2 - \lambda_2 I_n}{\lambda_1 - \lambda_2} e^{-\lambda_1 t} + \frac{A_2 - \lambda_1 I_n}{\lambda_2 - \lambda_1} e^{-\lambda_2 t}$$
$$= \begin{bmatrix} e^t & 0 \\ e^t - e^{2t} & e^{2t} \end{bmatrix} \quad (2.10)$$

from (2.7) we have the desired boundary conditions

$$x(0,1) = e^{-A_2 t_f} x_f = \begin{bmatrix} e^{t_f} & 0 \\ e^{t_f} - e^{2t_f} & e^{2t_f} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Bigg|_{t_f=1}$$
$$= \begin{bmatrix} 2e \\ 2e + e^2 \end{bmatrix} \quad (2.11)$$

and $x(t,0) = 0$, $\dot{x}(t,0) = 0$, $t \geq 0$.

The above conditions can be extended as follows.
From (2.1) for $i = 1$ we have

$$\dot{x}(t,2) = A_2 x(t,2) + F(t,1) \quad (2.12)$$

where

$$F(t,1) = A_0 x(t,1) + A_1 \dot{x}(t,1) . \qquad (2.13)$$

Substitution of (2.3) and (2.5) for $F(t,0) = 0$ into (2.13) yields

$$F(t,1) = (A_0 + A_1 A_2) x(t,1) = (A_0 + A_1 A_2) e^{A_2 t} x(0,1) . \ (2.14)$$

Assuming $x(0,1) = 0$ we obtain $F(t,1) = 0$ and from (2.12)

$$x(t,2) = e^{A_2 t} x(0,2) \qquad (2.15)$$

Continuing this procedure for $i = 2,\ldots,q-1$ we obtain the following theorem, which is an extension of Theorem 2.1.

Theorem 2.3. The general model (2.1) is always pointwise complete at the point $(t_f,\ q)$, $t_f > 0$, $q \in N = \{1,2,\ldots\}$ for any matrices A_k, $k = 0,1,2$.

Theorem 2.2 can be also extended for any point (t_f, q).

3. POINTWISE DEGENERACY AND POINTWISE DEGENERACY OF THE POSITIVE GENERAL MODEL

Definition 3.1. The model (2.1) is called positive if $x(t,i) \in \Re_+^n$, $t \in \Re_+$, $i \in Z_+$ for any boundary conditions

$$x_{t0} \in \Re_+^n, \ x_{t1} \in \Re_+^n, \ t \in \Re_+, \ x_i \in \Re_+^n, \ i \in Z_+ \ (3.1)$$

Theorem 3.1. The general model (2.1) is positive if and only if

$$A_2 \in M_n \qquad (3.2a)$$

$$A_0, A_1 \in \Re_+^{n \times n}, \ A = A_0 + A_1 A_2 \in \Re_+^{n \times n} \qquad (3.2b)$$

where M_n is the set of $n \times n$ Metzler matrices (with nonnegative off-diagonal entries).

Proof. Necessity. Necessity of $A_0 \in \Re_+^{n \times n}$ and $A_1 \in \Re_+^{n \times n}$ follows immediately from (2.4) since $F(t,0) \in \Re_+^n$, $t \in \Re_+$ and x_{t0}, x_{t1} are arbitrary. From (2.5) it follows that $A_2 \in M_n$ since $e^{A_2 t} \in \Re_+^{n \times n}$ only if A_2 is a Metzler matrix, $x(t,1) \in \Re_+^n$, $t \in \Re_+$ and $x(0,1)$ is arbitrary. From (2.12) it follows that $F(t,1) \in \Re_+^n$, $t \in \Re_+$ for any $x(0,1) \in \Re_+^n$ only if $A = A_0 + A_1 A_2 \in \Re_+^{n \times n}$. The proof of sufficiency is similar to the one given in (Kaczorek 2002: pp.255). □

Definition 3.2. The positive general model (2.1) is called pointwise complete at the point (t_f, q) if for every final state $x_f \in \Re_+^n$ there exist boundary conditions (3.1) such that

$$x(t_f, q) = x_f, \ t_f > 0, \ q \in N = \{1,2,\ldots\} . \qquad (3.3)$$

It is assumed that $x_{t0} = 0$ and $x_{t1} = 0$ for $t \in \Re_+$.

Theorem 3.2. The positive general model (2.1) is pointwise complete at the point $(t_f,\ 1)$ if and only if the matrix A_2 is diagonal.

Proof. In a similar way as in proof of Theorem 2.1 we may obtain the equation (2.7). It is well-known (Kaczorek 2002) that $e^{A_2 t} \in \Re_+^{n \times n}$, $t \in \Re_+$ if and only if A_2 is a Metzler matrix. Hence $e^{-A_2 t_f} \in \Re_+^{n \times n}$ if and only if A_2 is a diagonal matrix. In this case for arbitrary $x_f \in \Re_+^n$ if and only if $x(0,1) \in \Re_+^n$. □

In a similar way as for standard general model we can prove the following theorem.

Theorem 3.3. The positive general model (2.1) is pointwise complete at the point $(t_f,\ q)$ $t_f > 0$, $q \in N = \{1,2,\ldots\}$ if and only if the matrix A_2 is diagonal.

From Theorem 3.3 we have the following corollary.

Corollary 3.1. The pointwise completeness of the positive general model (2.1) is independent of the matrices A_0 and A_1 of the model.

Definition 3.3. The positive general model (2.1) is called pointwise degenerated at the point (t_f, q) if there exists at least one final state $x_f \in \Re_+^n$ such that $x(t_f, q) \neq x_f$ for all $x(0,i) \in \Re_+^n$ and $x(t,0) = 0$, $\dot{x}(t,0) = 0$, $t \in \Re_+$.

Theorem 3.4. The positive general model (2.1) is pointwise degenerated at the point (t_f, q) if the matrix $A_2 \in M_n$ is not diagonal.

Proof. In a similar way as in proof of Theorem 2.1 we may obtain the equality (2.7) which can be satisfied for $x_f \in \Re_+^n$ and $x(0,1) \in \Re_+^n$ if and only if the Metzler matrix A_2 is diagonal. The proof for $q > 1$ is similar. □

These considerations can be easily extended for $x(0,i) \in \Re_+^n$, $x(t,0) \in \Re_+^n$ and $\dot{x}(t,0) \in \Re_+^n$, $t \in \Re_+$.

Example 3.1. Consider the general model (2.1) with the matrices

$$A_0 = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}. \quad (3.4)$$

The model is positive since the matrices A_0 and A_1 have nonnegative entries and

$$\begin{aligned} A &= A_0 + A_1 A_2 \\ &= \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \Re_+^{2 \times 2} \end{aligned} \quad (3.5)$$

The matrix A_2 is diagonal and the positive model with (3.4) by Theorem 3.2 is pointwise complete at the point $(t_f,\ 1)$, $t_f \geq 0$. Using (2.7) we obtain

$$x(0,1) = e^{-A_2 t_f} x_f = \begin{bmatrix} e^{t_f} & 0 \\ 0 & e^{2 t_f} \end{bmatrix} x_f \in \Re_+^2 \quad (3.6)$$

for any $x_f \in \Re_+^2$ and $t_f \in \Re_+$.

Example 3.2. Consider the general model (2.1) with the matrices (2.9). The model is positive since A_0 and A_1 have nonnegative entries, $A_2 \in M_n$ and

$$A = A_0 + A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \in \Re_+^{2 \times 2}. \qquad (3.7)$$

Let $x_f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Using (2.10) and (2.7) we obtain the vector

$$x(0,1) = e^{-A_2 t_f} x_f = \begin{bmatrix} e^{t_f} & 0 \\ e^{t_f} - e^{2t_f} & e^{2t_f} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{t_f} \\ e^{t_f} - e^{2t_f} \end{bmatrix} (3.8)$$

with nonnegative second component for $t_f > 0$. Therefore, the model is pointwise degenerated at the point $(t_f, 1)$. The same result follows from Theorem 3.4 since the matrix A_2 is not diagonal. Note that the vector $x(0,1)$ given by (2.11) for $x_f = [2 \quad 3]^T$ has positive components.

4. CONCLUDING REMARKS

The pointwise completeness and pointwise degeneracy of the standard and positive hybrid linear systems described by the general model have been addressed. Necessary and sufficient conditions for the pointwise completeness and pointwise degeneracy have been established. It has been shown that the standard general model (2.1) is always pointwise complete at the point (t_f, q) for any t_f and $q \geq 1$ and it is not pointwise degenerated at any point. Necessary and sufficient conditions for the positivity of general model (2.1) have been established. The positive general model is pointwise complete at the point (t_f, q) for $t > 0$, $q \geq 1$ if and only if the Metzler matrix A_2 of the model is diagonal. The considerations have been illustrated by numerical examples. These considerations can be extended for linear hybrid systems with delays. An extension of these considerations for fractional hybrid systems is an open problem.

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AUTHORS BIOGRAPHY

Tadeusz Kaczorek, born 27.04.1932 in Poland, received the MSc., PhD and DSc degrees from Electrical Engineering of Warsaw University of Technology in 1956, 1962 and 1964, respectively. In the period 1968 - 69 he was the dean of Electrical Engineering Faculty and in the period 1970 - 73 he was the prorektor of Warsaw University of Technology. Since 1971 he has been professor and since 1974 full professor at Warsaw University of Technology. In 1986 he was elected a corresponding member and in 1996 full member of Polish Academy of Sciences. In the period 1988 - 1991 he was the director of the Research Centre of Polish Academy of Sciences in Rome. In June 1999 he was elected the full member of the Academy of Engineering in Poland. In May 2004 he was elected the honorary member of the Hungarian Academy of Sciences. He was awarded by the University of Zielona Gora (2002) by the title doctor honoris causa, the Technical University of Lublin (2004), the Technical University of Szczecin (2004), Warsaw University of Technology (2004), Bialystok Technical University (2008), Lodz Technical University (2009) and Opole Technical University (2009).

His research interests cover the theory of systems and the automatic control systems theory, specially, singular multidimensional systems, positive multidimensional systems and singular positive 1D and 2D systems. He has initiated the research in the field of singular 2D, positive 2D linear systems and positive fractional 1D and 2D systems. He has published 23 books (6 in English) and over 940 scientific papers.

He supervised 69 Ph.D. theses. He is editor-in-chief of Bulletin of the Polish Academy of Sciences, Techn. Sciences and editorial member of about ten international journals.