NON-UNIFORM FINITE SEGMENT REPRESENTATION AND REDUCTION OF A CANTILEVER BEAM MODEL

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ABSTRACT

The cantilever beam is a component widely used in numerous engineering systems with its geometric and material properties varying depending on the application. Calculating the dynamic behavior of a cantilever beam is a challenging task since the critical physical phenomena and interactions vary significantly based on the geometry of the beam. There exist a number of theories that can be used to model the transverse motion of a cantilever beam of which one of the most commonly used is the Timoshenko beam theory. The Timoshenko theory is relatively more complex, than other theories used to model transverse motion of beams, however it is frequently used because it gives accurate predictions over a wider range of beam parameters. When a beam is part of a system as an individual component, then the model is developed using a finite segment approach since the beam can be easily connected with other lumped parameters components. In the case of a finite segment representation, it is critical to know the number of segments that the beam is divided into, in order to have accurate predictions of the dynamic behavior. Previous work by the author proposed a systematic procedure for addressing the issues of number of segments and physical phenomena to include in a dynamic model of a cantilever beam. It is the objective of this paper to extent that work by using variable segment length in order to further reduce the complexity of the model.

The paper presents a new approach for developing a reduced model of a cantilever beam. The beam is discretized through the finite segment approach, however, the segments do not have the same length across the axis of the beam. More refined division is used at the areas with high dynamical behavior and a coarser division is used elsewhere. The model is developed using the bond graph formulation. The activity metric and the Model Order Reduction Algorithm is then used to determine which physical phenomena (inertial and compliant) need to be included in the model in order to have accurate predictions of the dynamic behavior. An illustrative example is provided to demonstrate the new methodology.

Keywords: Finite segment modeling, cantilever beam, model reduction, activity metric.

1. INTRODUCTION

Modeling and simulation have yet to achieve wide utilization as commonplace engineering tools. One reason for this is that current modeling and simulation techniques are inadequate. Specifically, a major disadvantage is that they require sophisticated users who are often not domain experts and thus lack the ability to effectively utilize the model and simulation tools to uncover the important design trade-offs. Another drawback is that models are often large and complicated with many parameters, making the physical interpretation of the model outputs, even by domain experts, difficult. This is particularly true when "unnecessary" features are included in the model.

A variety of algorithms have been developed and implemented to help automate the production of proper models of dynamic systems. Wilson and Stein (1995) developed MODA (Model Order Deduction Algorithm) that deduces the required system model complexity from subsystem models of variable complexity using a frequency-based metric. They also defined proper models as the models with physically meaningful states and parameters that are of necessary but sufficient complexity to meet the engineering and accuracy objectives. Additional work on deduction algorithms for generating proper models in an automated fashion has been reported by Ferris and Stein (1995), Ferris et al. (1998), and Walker et al. (1996). The above algorithms have also been implemented in an automated modeling computer environment (Stein and Louca, 1996).

In an attempt to overcome the limitations of the frequency-based metrics, Louca et al. (2010) introduced a new model reduction technique that also generates proper models. This approach uses an energy-based metric (element activity) that in general, can be applied to nonlinear systems, and considers the importance of all energetic elements (generalized inductance, capacitance and resistance). The contribution of each energy element in the model is ranked according to the activity metric under specific excitation. Elements with small contributions are eliminated in order to produce a reduced model using a systematic methodology called Model Order Reduction Algorithm (MORA). The activity metric was also used as a basis for even further reduction, through partitioning the model into smaller and decoupled submodels (Rideout et al. 2007).

Such modeling approaches should be able to handle real mechanical systems that typically include distributed parameter (continuous) components, e.g. rods, beams, plates, etc. Frequently, modeling objectives and assumptions allow the lumping of continuous component properties into ideal energy elements that lead to a

dynamic model described by a set of ordinary differential equations. However, when property lumping is not acceptable, modeling of a continuous component requires a different approach since its inertial, compliance and resistive properties are spatially distributed and cannot be lumped into single equivalent elements. The dynamic behavior of continuous components is thus described by partial differential equations with derivatives in both time and space. Another approach that is considered in this work is the modeling of a continuous component with finite segments that are spatially distributed. The accuracy is a function of the number of segments for this approximation. The model accuracy improves as the number of segments increases. Model accuracy and the required number of segments can be addressed using a frequency-based metric (Ferris et al., 1998).

Beyond physical-based modeling, modal decomposition is also used to model and analyze continuous and discrete systems (Meirovitch, 1967). One of the advantages of modal decomposition is the ability to easily adjust (i.e., reduce) model complexity since all modes are orthogonal to each other. The reduction of such modal decomposition models is mostly based on frequency, and the user defined frequency range of interest (FROI) determines the frequencies that are important for a specific scenario. In this case, modes with frequencies within the FROI are retained in the reduced model and modes outside this range are eliminated. As expected, mode truncation introduces error in the predictions that can be measured and adjusted based on the accuracy requirements (Li and Gunter, 1981; Liu et al., 2000).

In the case of modeling continuous components (beam), element activity can be used to address model complexity in terms of number of segments and physical phenomena to be included in the model (Louca, 2015). The optimum number of segments can be defined using the input activity and the important physical phenomena using the activity of each energy element (inertial, compliant, and resistive). This optimum complexity was identified using a uniform division of the beam into segments of equal length. That analysis indicated that the segments towards the ends of the beam have higher activity due the higher inertial or compliant effects, leading to a reduced model with accurate predictions. Based on this observation the work in this paper introduces a new procedure for modeling continuous components, e.g., cantilever beam, in which the beam is not divided into equal length segments. Shorter segments are used towards the ends of the beam with the goal to have an even lower complexity of the reduced model.

This paper is organized as follows: first, background about the energy-based activity metric is provided, along with the reduction algorithm and the closed-form expressions for steady state activities. Next, the equation formulation for a finite segment cantilever beam with non-uniform segment lengths is presented. Then the complexity of a cantilever beam is analyzed using

MORA. Finally, discussion and conclusions are given in the last section.

2. BACKGROUND

The original work on the energy-based metric for model reduction is briefly described here for convenience. The activity metric has been previously formulated for systems with nonlinearities in both the element constitutive laws and junction structure (Louca and Yildir, 2006; Louca et al., 2010). In this work, the activity metric is applied to linear systems for which analytical expressions for the activity can be derived, and therefore, avoid the use of numerical time integration that could be cumbersome (Louca and Stein, 2009). The analysis is further simplified if, in addition to the linearity assumption, the system is assumed to have a single sinusoidal excitation, and only the steady state response is examined. These assumptions are motivated from Fourier analysis where an arbitrary function can be decomposed into a series of harmonics. Using this frequency decomposition, the activity analysis can be performed as a function of frequency in order to study the frequency dependency of element activity.

2.1. Element Activity for Linear Systems

A measure of the power response of a dynamic system, which has physical meaning and a simple definition, is used to develop the modeling metric, element activity (or simply "activity"). Element activity, A, is defined for each energy element as:

$$A = \int_0^\tau |P(t)| dt \qquad (1)$$

where $P(t)$ is the element power and τ is the time over which the model has to predict the system behavior. The activity has units of energy, representing the amount of energy that flows in and out of the element over the given time τ. The energy that flows in and out of an element is a measure of how active this element is (how much energy passes through it), and consequently the quantity in Eq. (1) is termed activity. Activity can be defined independent of the energy domain, type of energy element or nonlinearities.

The activity is calculated for each energy element based on the system response. In the case that the system is modeled using a bond graph formulation, the state equations are derived using the multi-port bond graph representation (Borutzky, 2004; Brown, 2006; Karnopp et al., 2006; Rosenberg and Karnopp, 1983). In addition, when a system has a single input and linear junction structure and constitutive laws, the state equations are linear time invariant and have the following general form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \qquad (2)$$

where, $\mathbf{A} \in \mathbb{R}^{m \times m}, \mathbf{b} \in \mathbb{R}^m$ are the state space matrices, $\mathbf{x} \in \mathbb{R}^m$ is the state vector, $u \in \mathbb{R}$ is the input, and m is the number of independent state variables.

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For the above system, appropriate outputs are defined in order to calculate the power of each energy element in the model using the constitutive law of each element. For convenience, the outputs are selected to be generalized flow, effort, and flow for inertial, compliant, and resistive elements, respectively. The dual effort or flow needed for calculating the power is derived from the output variables and constitutive laws. The output vector for this set of variables has the form:

$$\mathbf{y} = \left\{ \begin{array}{c} \mathbf{f}_I \\ \hline \mathbf{e}_C \\ \hline \mathbf{f}_R \end{array} \right\} \qquad (3)$$

where $\mathbf{y} \in \mathbb{R}^k$ and $\mathbf{f}_I \in \mathbb{R}^{k_I}, \mathbf{e}_C \in \mathbb{R}^{k_C}$, and $\mathbf{f}_R \in \mathbb{R}^{k_R}$. The variables k_I, k_C, and k_R represent the number of inertial, compliant, and resistive elements, respectively. The total number of energy elements is $k = k_I + k_C + k_R$. Using the output variables set in Eq. (3), the output equations can be written as:

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{d}u \qquad (4)$$

where $\mathbf{C} \in \mathbb{R}^{k \times m}, \mathbf{d} \in \mathbb{R}^k$ are the output state space matrices.

Given this set of output variables the missing efforts or flows, needed for calculating the element power, are computed from the linear constitutive laws of each type of energy element as shown below:

$$\begin{aligned} \mathbf{I}: & \quad p_I = r_I f_I \Leftrightarrow e_I = \dot{p}_I = r_I \dot{f}_I \\ \mathbf{C}: & \quad q_C = r_C e_C \Leftrightarrow f_C = \dot{q}_C = r_C \dot{e}_C \\ \mathbf{R}: & \quad e_R = r_R f_R \end{aligned} \qquad (5)$$

where r_I, r_C, r_R are known constants representing the linear constitutive law coefficients of inductance, compliance and resistance, respectively. For more compact expressions a vector, $\mathbf{r} \in \mathbb{R}^k$, with all the linear constitutive law coefficients is introduced as shown below:

$$\mathbf{r} = \left\{ \begin{array}{c} \mathbf{r}_I \\ \hline \mathbf{r}_C \\ \hline \mathbf{r}_R \end{array} \right\} \qquad (6)$$

where $\mathbf{r}_I \in \mathbb{R}^{k_I}$, $\mathbf{r}_C \in \mathbb{R}^{k_C}$, and $\mathbf{r}_R \in \mathbb{R}^{k_R}$ are the constant constitutive law coefficients.

Finally, the power needed for calculating the activity of each element, as defined in Eq. (1), is computed as the product of generalized effort and flow, and by using Eq. (5) the following expressions for the activity are derived:

$$A_i = \int_0^\tau |\mathbb{P}_i| = r_i \int_0^\tau |y_i \dot{y}_i| \, dt, \quad i = 1, \ldots, k_I + k_C$$

$$A_i = \int_0^\tau |\mathbb{P}_i| = r_i \int_0^\tau |y_i^2| \, dt, \quad i = k_I + k_C + 1, \ldots, k \qquad (7)$$

2.2. Activity for Single Harmonic Excitation

The time response of the output vector, $\mathbf{y}(t)$, in Eq. (3) is required in order to complete the calculation of the element power. For the purposes of this work, the excitation is assumed to be a single harmonic given by:

$$u(t) = U \sin(\omega t) \qquad (8)$$

where $U \in \mathbb{R}$ is the amplitude of the excitation and ω is the excitation frequency. The steady state response of the linear system in Eq. (2) and (4), and for the above excitation, is easily calculated using linear system theory. This gives the following closed form expression:

$$y_i(t, \omega) = U Y_i(\omega) \cdot \sin(\omega t + \varphi_i(\omega)), \ i = 1, \ldots, k \qquad (9)$$

where $Y_i(\omega)$ and $\varphi_i(\omega)$ are the steady state amplitude and phase shift, respectively, which can be easily calculated from the state space matrices using linear systems theory.

Within the context of this analysis, the output $y_i(t, \omega)$ in Eq. (9) is either an effort or a flow that is used to calculate the activity of each element in Eq. (7). Finally, the activity can be calculated by Eq. (1), but first the upper bound of this integral must be specified. For this case, the steady state and periodicity of the response are exploited. A periodic function repeats itself every T seconds, and therefore, a single period of this function contains the required information about the response. Thus, the upper bound of the integral is set to one period of the excitation, $\tau = T = 2\pi/\omega$. Therefore, the steady state activity for the energy storage elements is given by:

$$A_i^{ss}(\omega) = r_i \int_0^T |y_i \dot{y}_i| \, dt = 2 r_i U^2 Y_i^2(\omega) \qquad (10)$$

and for energy dissipation elements by:

$$A_i^{ss}(\omega) = r_i \int_0^T |y_i^2| \, dt = \frac{\pi r_i U^2 Y_i^2(\omega)}{\omega} \qquad (11)$$

The above simple closed form expressions can be used to calculate the activity of all energy elements for a given single harmonic excitation. The superscript 'ss' in Eq. (10)-(11) denotes the activity under a steady state harmonic response.

2.3. Activity Index and MORA

The activity as defined in Eq. (1) is a measure of the absolute importance of an element as it represents the amount of energy that flows through the element over a given time period. In order to obtain a relative measure of the importance, the element activity is compared to a quantity that represents the "overall activity" of the system. This "overall activity" is defined as the sum of all the element activities of the system and it is termed total activity, A^{Total}. Thus a normalized measure of element importance, called the element activity index or just activity index, is defined as:

$$AI_i^{ss}(\omega) = \frac{A_i(\omega)}{A^{Total}(\omega)} = \frac{A_i^{ss}(\omega)}{\sum_{i=1}^{k} A_i^{ss}(\omega)} \qquad (12)$$

where A_i^{ss} is the activity of the ith element given by Eq. (10) and (11). The activity index, $AI_i^{ss}(\omega)$, is calculated for each element and it represents the portion of the total system energy that flows through an element.

With the activity index defined as a relative metric for addressing element importance, the Model Order Reduction Algorithm (MORA) is constructed. The first step of MORA is to calculate the activity index for each element in the system for a given system excitation and initial conditions. Next, the activity indices are sorted to identify the elements with high activity (most important) and low activity (least important). With the activity indices sorted, the model reduction proceeds given the desired engineering specifications. These specifications are defined by the modeler who then converts them into a threshold β of the total activity (e.g., 99%) that he or she wants to include in the reduced model. This threshold defines the borderline between the retained and eliminated model elements. The elimination process is shown in Figure 1 where the sorted activity indices are summed starting from the most important element until the specified threshold is reached. The element which, when included, increments the cumulative activity above the threshold, is the last element to be included in the reduced model. The elements that are above this threshold are removed from the model.
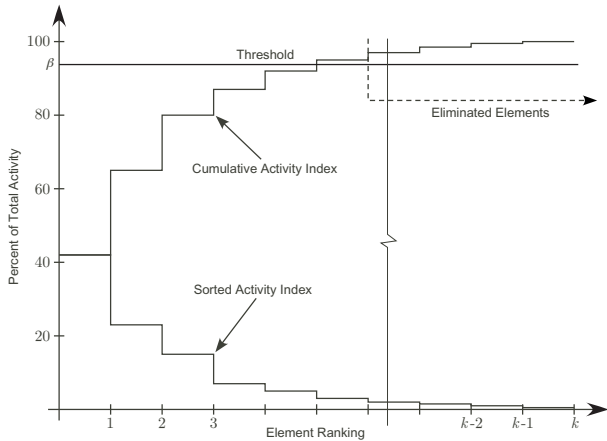

Figure 1: Activity index sorting and elimination.

3. TIMOSHENKO BEAM MODEL

Models of continuous systems can be developed using solid mechanics techniques, which lead to Partial Differential Equations (PDE) with derivatives in both space and time (Bauchau and Craig, 2009; Genta, 2009; van Rensburg and van der Merwe, 2006; Li, 2008). The continuous cantilever beam used in this work is shown in Figure 2, where its transverse motion is considered when excited with a vertical load at its free end. The motion of a given cross section, $w(x,t)$ and $\varphi(x,t)$, from its undeformed state varies with time and location thus having PDEs describing its motion. Note that due to the

rotation φ, a cross section does not remain normal to the neutral axis according to the Timoshenko beam theory that is used in this work. One method for solving these PDEs is separation of variables, which produces a modal expansion solution (Meirovitch, 1967). An analysis of the advantages and disadvantages of this approach is beyond the scope of this work, however, it is safe to say that the solution of PDEs is more cumbersome than the solution of ordinary differential equations that describe the behavior of lumped parameters system.
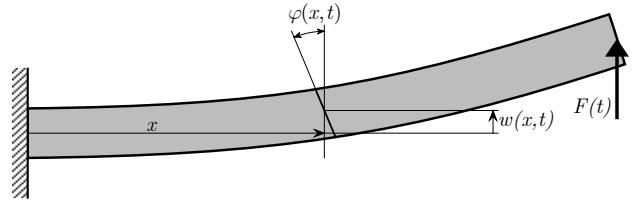

Figure 2: Cantilever beam transverse vibration.

A different approach for modeling the transverse vibration of a cantilever beam is to divide it into segments of equal length. This approach is motivated by the procedure for deriving the PDEs describing the motion of a beam. Each of these segments has linear inertial and compliant properties that can be determined from solid mechanics theory. Shear effects and rotational inertial effects are also considered, which results in a more generic model that is valid for a larger range of geometric parameters. This is known as the Timoshenko beam model, which is usually used for non-slender beams in order to get accurate model predictions. The use of this more complex model using the Timoshenko beam theory is also mandated from the use of MORA in the process of determining the appropriate model complexity. In this approach the most complex model is first developed, and then MORA is used to identify what is actually needed in order to reach a reduced model with accurate predictions.

3.1. Non-uniform Finite Segments

The ideal physical model under these assumptions is shown in Figure 3 where the beam is divided into n segments. This model approaches the partial differential equations of the continuous system, as the number of segments approaches infinity. Previous work studied modeling and model reduction of a cantilever beam when it is uniformly divided and the length of segments are equal, i.e., $\Delta x = L/n$. The number of segments and physical phenomena to be included in each segment, while accurately predicting the dynamic behavior, were identified using activity (Louca, 2014; Louca, 2015).
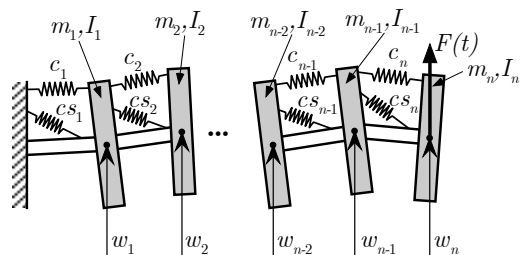

Figure 3: Ideal physical model of a Timoshenko beam.

For the purposes of this work, the segments are assumed not to have equal length based on results of previous work. The activity analysis, for equal length segments, identified as high activity and important elements the ones near the ends of the beam. The elements near the fixed end (left) have high activity due to the high bending and thus high compliant effects. On the other hand, elements near the free end (right) have high activity due to the high motion and thus high inertial effects. On the contrary, element around the middle have lower activities since they have lower bending and motion.

Therefore, a different division of the beam is proposed in order to address the localization of the dynamic effects. Shorter segments are used towards the ends, while longer segments are used around the middle of the beam. The transition of segment length from the ends towards the middle is selected to be linear for simplicity as shown in Figure 4. In addition, segment length is assumed to be symmetric. The variable segment length for the first half (left) of the beam is therefore given by the linear equation:

$$\Delta x_i = a + b(i-1), \qquad i = 1,\ldots,n/2 \qquad (13)$$

where a and b are constants defining the initial length and rate of increment, respectively.
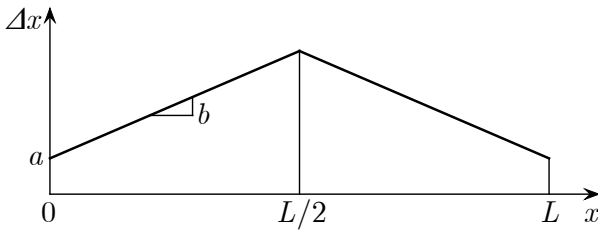

Figure 4: Linear increase in segment length.

In order to have an increase in segment length, and not a decrease, the constant a has to be smaller than the segment length for uniform division, i.e., $a < L/n$. Given a value for a, constant b is calculated such that:

$$\sum_{i=1}^{n/2} \Delta x_i = \frac{L}{2} \qquad (14)$$

Constant b is calculated by substituting Eq. (13) into Eq. (14). Solution of the resulting equation for b gives:

$$b = \frac{4(L - an)}{n(n-2)} \qquad (15)$$

The slope of the segment length is increasing as the initial segment length, a, decreases. Thus this parameter controls the segment length variation along the beam.

Similarly the length of the segments for the second half (right) of the beam is given by the linear equation in Eq. (16). This equation provides a linear decrease in segment length and the lengths of the last and first segment are equal, i.e., $\Delta x_1 = \Delta x_n = a$. A sample

division of a beam into non-uniform but linearly varying segments is given in Figure 5, where $a = 0.4\,L/n$ and $n = 20$. Segment lengths are symmetric with shorter segments toward the ends.

$$\Delta x_i = \Delta x_{n/2} - b(i - 1 - n/2), \, i = n/2 + 1,\ldots,n \, (16)$$
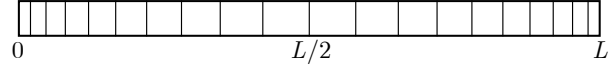

Figure 5: Non-uniform segment lengths.

The location of the each segment, x_i, along the axis of the beam is calculated by the summation of the previous segment lengths as given by Eq. (17) and the location of the last segment is equal to the length of the beam, i.e., $x_n = L$.

$$x_i = \sum_{j=1}^{i} \Delta x_j, \quad i = 1,\ldots,n \qquad (17)$$

Given the segment length the constitutive law parameters of the energy storage elements are calculated, and the beam is assumed to have density ρ, Young's modulus E, shear modulus G, length L, cross sectional area A and cross sectional moment of inertia I. Given these physical parameters of the beam, the element parameters in the above linear model are given by the expressions below:

$$
\begin{aligned}
m_i &= \rho A \Delta x_i, \quad i = 1,\ldots,n \\
I_i &= \rho I \Delta x_i, \\
c_i &= \frac{\Delta x_i}{EI} \\
cs_i &= \frac{\Delta x_i}{\kappa G A}
\end{aligned}
\qquad (18)
$$

where κ is a dimensionless constant that accounts for the non-uniform distribution of the shear stress and depends on the shape of the cross-section. The inertial parameters m_i and I_i represent the linear and rotational inertia of each segment, respectively. The parameters c_i and cs_i represent the bending and shear compliance between two segments, respectively. These parameters are not equal across segments due to the different length of each segment. The beam is assumed to have no energy losses therefore there are no damping elements in the model. These parameters are used to define the parameter vector as defined in Eq. (6).

For developing the dynamic equations, the bond graph formulation is used. Bond graphs provide the power topography of the system and it is a natural selection for implementing the power-based activity metric. The bond graph model of the ideal physical model as shown in Figure 3 is developed and given in Figure 6. The bond graph has $4n$ independent state variables since each segment is modeled by 4 independent energy storage elements and its state vector has the form $\mathbf{x} = \{p_1,\ldots,p_n,p_{I1},\ldots,p_{In},q_1,\ldots,q_n,q_{s1},\ldots,q_{sn}\}^T$. The transverse velocity of the each mass, v_i, represents the

velocity at a given location of the continuous beam and Eq. (19) expresses the relation between the discrete and continuous variables. The other kinematic variable of the model, ω_i, is the rotation at a given location and its relation to the continuous variable is given in Eq. (20).

$$v_i(t) = \dot{w}_i = \dot{w}(x_i, t) \qquad (19)$$

$$\omega_i(t) = \dot{\varphi}_i = \dot{\varphi}(x_i, t) \qquad (20)$$

For easy calculation of the state equations and the output equations that are required for calculating power, the equations are derived using the multi-port approach (Rosenberg, 1971). According to this approach, the state space and input matrices are given by:

$$\mathbf{A} = \mathbf{J}_{SS}\mathbf{S}, \quad \mathbf{b} = \mathbf{J}_{SU} \qquad (21)$$

The output matrices, as defined in Eq. (4), that are required for calculating the power flow into the energy elements are given by:

$$\mathbf{C} = \mathbf{S}, \quad \mathbf{d} = \mathbf{0} \qquad (22)$$

The output vector according to the analysis in the previous section is given by $\mathbf{y} = \{f_1, \ldots, f_{2n}, e_1, \ldots, e_{2n}\}^T$. The dimensions of the state space matrices are $m = 4n$ and $k = 4n$. Based on this set of state variables, the junction structure matrices, $\mathbf{J}_{SS}$ and $\mathbf{J}_{SU}$, are derived and given in the Appendix.

For the above model with n segments the steady-state response is first calculated using Eq. (9) and based on the state space equations in Eq. (21)-(22). Then the element activity is calculated from Eq. (10) and (11), which gives the following expression for the energy storage elements of the model:

$$A_i^{ss}(\omega) = 2r_iU^2Y_i^2(\omega), \quad i = 1, \ldots, 4n \qquad (23)$$

The above analysis enables the calculation of the element activity for a given single harmonic excitation. The activity index that is used by MORA is independent of the excitation amplitude, as shown in Eq. (12), and therefore can be set to an arbitrary value, e.g., set to one (1) for simplicity. Model complexity and which physical phenomena need to be included can be determined given the element activity in Eq. (23) and MORA. The complexity of the beam is investigated in the next section in order to identify the significant elements based on beam length and element location. A series of analyses is performed in order to get more insight into the beam dynamics under different scenarios.

4. BEAM COMPLEXITY BASED ON ACTIVITY

The activity metric and MORA is applied to a steel cantilever beam with parameters $\rho = 7860$ kg/m^3, $E = 210$ GPa, $G = 80$ GPa, $A = 3\times10^{-3}$ m^2, $I = 2.5\times10^{-6}$ m^4, $\kappa = 0.85$. The length of the beam is varied, $L = 0.2$-2 m, in order to study the variation of element significance. The methodology is easy and computationally inexpensive to implement due to the simple and closed form expressions used for calculating the state space matrices, frequency response and activity.

First, the beam length is set to 2 m and the length coefficient $a = 0.1L/n$. The number of segments is set to $n = 30$ and therefore there are a total of 120 energy storage elements modeling the beam. The modeling target is to accurately predict static behavior to low frequency dynamics, thus the excitation frequency is set to 95% of the first natural frequency (122.68 rad/s).

The results of the activity analysis using Eq. (23) and under these assumptions are shown in Figure 7 where the activity index of all 120 elements is shown. Element numbers 1-30 represent the activity index of the linear inertia (m_i) and 31-60 the activity index of the rotational inertia (I_i) of each segment. Next, element numbers 61-90 and 91-120 represent the activity index of the bending (c_i) and shear (cs_i) compliance, respectively. For each range of elements the smallest numbers represent elements that are next to the fixed end of the beam. It is clear from the activity analysis that the most important elements are related to the linear inertia and the bending stiffness of the beam. On the contrary, the
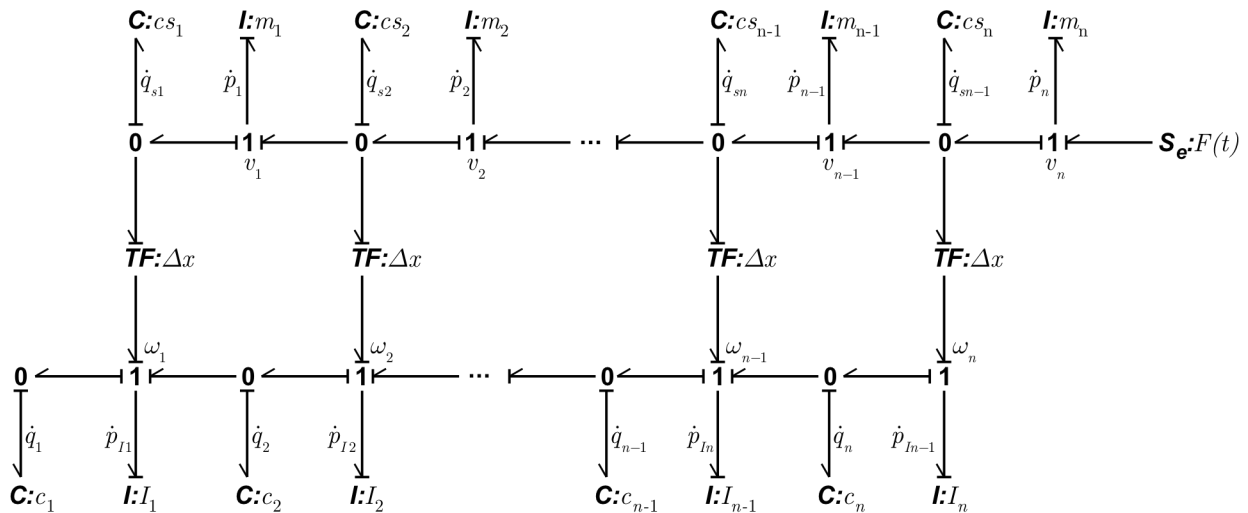

Figure 6: Bond graph model of a cantilever Timoshenko beam.

elements related with the rotary inertia and shear stiffness have very low activity and thus are insignificant under these conditions. The activity analysis agrees with common practice, in which a slender beam is modeled using the Euler-Bernoulli theory that neglects rotational inertia and shear stress effects.
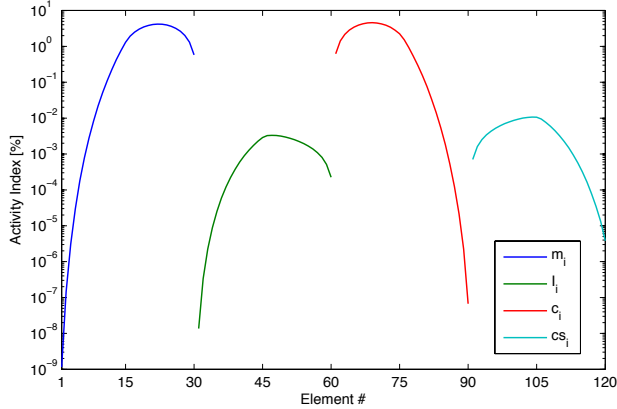

Figure 7: Element activity indices.

Model complexity is systematically addressed using MORA as it is described in Section 2.3. Elements are ranked according to their activity index as shown in Figure 8 where the sorted activity indices along with the cumulative activity index are plotted. According to activity analysis, 36 of the 120 elements account for almost 99% of the energy the flows through the model. This is a significant result verifying that unnecessary complexity is included in the model, however, the figure does not directly depicts the elements that are insignificant and could be eliminated from the model.
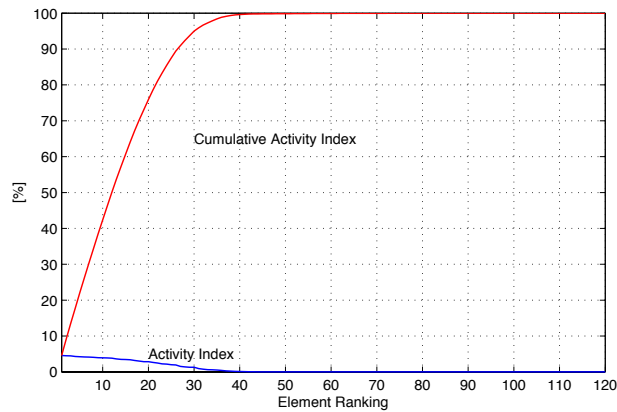

Figure 8: Element ranking for slender beam.

The important elements are next identified using MORA. Using a reduction threshold, $\beta = 99\%$, MORA identifies the elements that have a significant contribution to the system dynamic behavior. The results of this analysis are shown in Figure 9 where both the activity and elimination/inclusion in the reduced model are depicted. The '+' symbol identifies the elements with significant contribution and must be included, where the 'o' symbols identifies that an element is insignificant and must be eliminated from the full model in order to generate the reduced model. Out of the 120 elements only 37 are important and the remaining 83 can be

eliminated. More specifically, MORA identifies that all rotational inertia and shear stiffness elements must be eliminated from the model. Linear inertia elements that are close to the support have low activity and can be eliminated from the model, where inertia elements towards the free end of the beam have high activity and must be retained. The reverse is true for the bending stiffness elements, where the elements towards the free end can be eliminated and the ones near the support must be retained. More specifically, 18 of the linear inertia and 19 of the bending stiffness elements have high activity and must be included in the reduced model.
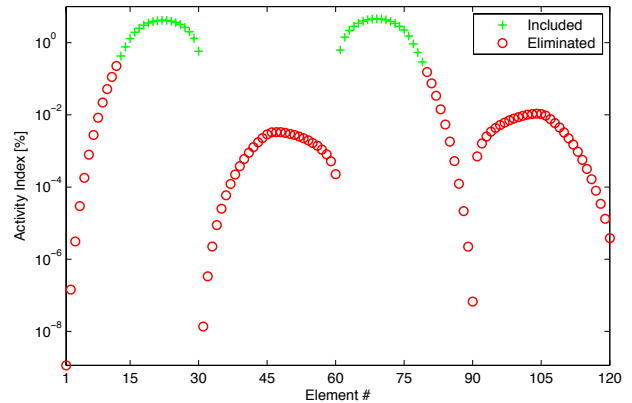

Figure 9: Model reduction for slender beam, $L = 2$ m.

The same reduction using MORA is performed with different beam lengths in order to study how element importance changes as the length is reduced. The reduction for a beam length of 1.0 m is shown in Figure 10. The same trend is observed for the elimination of linear inertia and bending stiffness elements. The activity index of all rotational inertia elements (31-60) is higher than before ($L = 2$ m) but still very low, and therefore, they are eliminated from the model. The activity of shear stiffness (91-120) also increases but still not significant to be included in the reduced model. A total of 40 elements are included in the reduced model with 20 linear inertia and 20 bending stiffness elements.
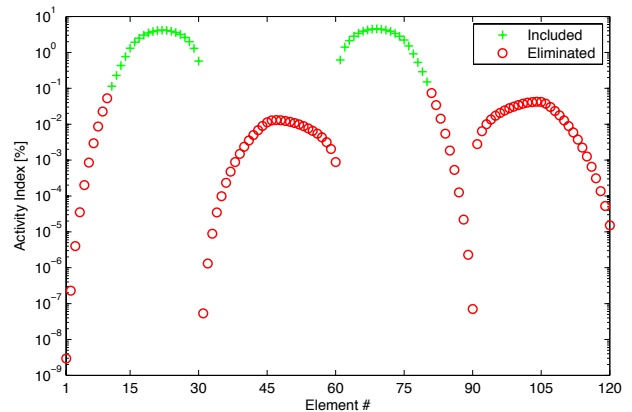

Figure 10: Model reduction for $L = 1.0$ m.

The beam length is further reduced to 0.2 m in order to examine if more elements become important. The activity index of the linear inertia and bending stiffness remains almost unchanged as shown in Figure 11.

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However, the activity index of the rotational inertia and shear stiffness is further increased such that some of them become important. More specifically the rotational inertia elements towards the free end are important and the ones near the fixed end are eliminated. A total of 74 elements out of 120 are included in the reduced model with 20 linear inertia, 13 rotational inertia, 20 bending stiffness and 21 shear stiffness elements.
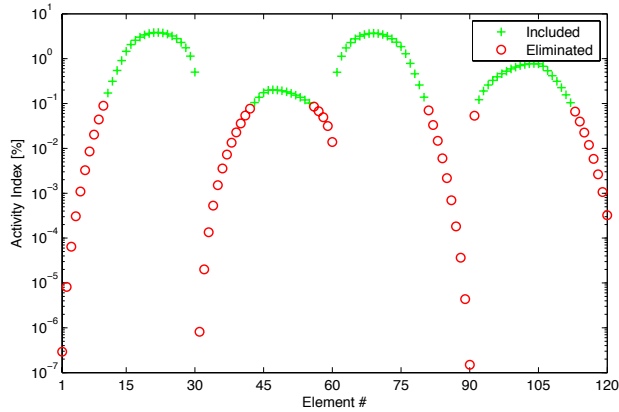

Figure 11: Model reduction for $L = 0.2$ m.

The variation of beam length showed that the total number of important elements increases as the beam length decreases. This variation is investigated in more detail by varying the beam length from 0.2 to 2 m with a step of 20 mm. The number of included linear and rotational inertia, and bending and shear stiffness is recorded along with the total number of elements. The results of this analysis are shown in Figure 12. The total number of elements is monotonically increasing as the beam length is decreased. The number of linear inertia and bending stiffness remains almost constant as the length changes. On the contrary, the number of shear stiffness elements is zero until about 0.9 m where it becomes important and starts increasing. Further reduction in length results in a monotonic increase in the number of included shear stiffness element. A similar behavior is observed for the number of the rotational inertia elements, however, they become important at a lower beam length of about 0.5 m.
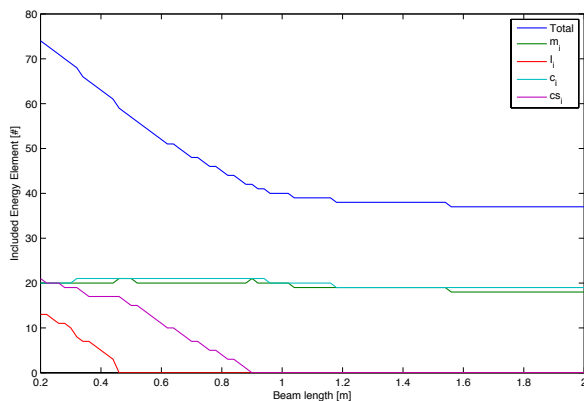

Figure 12: Model reduction for length variation.

The validity of the generated reduced models is verified by analyzing the accuracy of the model.

Specifically the steady state response amplitude for the velocity at the free end and the torque at the fixed end are calculated. The comparison is made with the corresponding response of the full model and over the range of beam lengths used before. The accuracy for both variables, as shown in Figure 13, varies as the beam length is changed, with averages around 85%. The variation in accuracy is due to the change of model complexity as different elements are added or removed in the reduced model according the activity metric. The accuracy of the first natural frequency is also studied and has similar trends, as the two outputs, but with an average accuracy around 99% (the plot is not included for brevity).
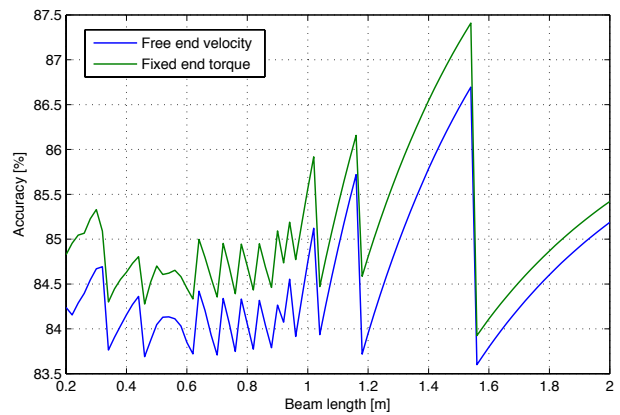

Figure 13: Model accuracy.

5. DISCUSSION AND CONCLUSIONS

A new methodology is developed that reduces the complexity of a cantilever beam model by providing more insight into the beam dynamic behavior at the same time. The proposed methodology provides a systematic modeling procedure for cantilever beams that are modeled through the finite segment approach. The beam is divided into segments in a non-uniform fashion in an attempt to achieve further reduction in the model size while maintaining the accuracy of the predictions. This is demonstrated through the example and reduced models have an average of 15% lower complexity (number of states) as compared to the complexity of reduced models with uniform segment lengths.

The results presented in this work are in agreement with the assumptions of beam theories, which propose that the Timoshenko beam model must be used for shorter rather than slender beams. The proposed methodology can be used when modeling beams, in order to decide which of the two models to use, Timoshenko or Euler-Bernoulli. In addition, the activity metric can refine the modeling assumptions by identifying what physical phenomena need to be included in each segment, i.e., linear and rotational inertia, bending and shear stiffness.

The number of segments is a significant parameter when it comes to modeling with the finite segment approach but it was considered constant in the analyses of the presented results. The methodology was also performed with various, lower and higher, number of

segments, however these results are not presented in this paper for brevity. The reduced models for different number of segments are identical with the ones presented in this work. The only difference is the actual number of included elements, as shown in Figure 12, however, the ratio of included elements to the total number of elements remains constant.

The number of segments of the reduced model is smaller when compared to the results of previous work, where the beam model was divided into segments of equal length (Louca, 2015). Thus, the proposed non-uniform segment length leads to an even higher reduction of model complexity.

The activity analysis is performed for a given single excitation frequency that is lower than the first natural frequency. This excitation is chosen since the model is expected to be used with low frequency excitations. A similar analysis with the one presented in this work can be performed for a higher frequency or range of frequencies in order to account for more realistic excitations. However, this procedure has to be formalized and this remains as an item for future research.

Because this work uses an energy-based modeling metric, it is convenient to use a model representation and formulation approach from which energy can be easily extracted/calculated. The bond graph approach explicitly presents the power topography of a dynamic system, and therefore, it is used in this work for calculating the necessary variables required for the power calculations. To be clear, the use of this methodology is not limited to systems represented by bond graphs. It can also be applied when the continuous system is modeled using any other modeling methodology, e.g., Lagrange's equations, Newton's Law, etc. However, in this case the calculation of power that is required for the proposed methodology might not be as trivial as using the bond graph formulation.

The results of this paper provide more insight into the nature of the reduced ordered models produced by MORA, and therefore, demonstrate that MORA is an even more useful tool than previously realized for the production of proper models of nonlinear systems. The activity metric effectively addresses the model complexity of distributed parameter components and in addition provides physical insight into the model.

APPENDIX: JUNCTION STRUCTURE MATRICES

$$\mathbf{J}_{SS} = \begin{bmatrix} \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & \mathbf{J}_1 \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & \mathbf{J}_1 & diag\left(\Delta x_i\right) \\ \mathbf{0}_{n \times n} & -\mathbf{J}_1^T & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ -\mathbf{J}_1^T & -diag\left(\Delta x_i\right) & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \end{bmatrix}$$

$$\mathbf{J}_{SU} = \left\{ \begin{array}{c} \mathbf{0}_{(n-1) \times 1} \\ 1 \\ \mathbf{0}_{3n \times 1} \end{array} \right\}$$

$$\mathbf{J}_1 = \begin{bmatrix} -1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} m_i \mathbf{I}_{n \times n} & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & I_i \mathbf{I}_{n \times n} & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & c_i \mathbf{I}_{n \times n} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} & cs_i \mathbf{I}_{n \times n} \end{bmatrix}^{-1}$$

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