ARMA MODEL BASED GPC

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ABSTRACT
In this paper an Auto-Regressive Moving Average (ARMA) model based Generalized Predictive Control (GPC) is presented. The controller derived from this method will automatically contain an integrator. The control law is obtained by minimizing a quadratic objective function. An analytical solution can be found in the absence of constraints. The presented method can deal with stable, unstable and non-minimum phase processes. A concept of suboptimal control is introduced in order to reduce calculating burden and refine dynamic aspects of a controlled system in case of need. It is straightforward to achieve the double integral action which is required in some industrial processes. Furthermore, it is very easy to incorporate the terms T and S if it is viewed as a classical RST controller. For this purpose, a relationship between GPC and RST controllers is also presented. Some simple examples and numerical analyses of particular cases are given.

Keywords: generalized predictive control, model predictive control, auto-regressive moving average, integral action

1. INTRODUCTION

As well known the MPC presents many advantages such as the availability to control a process with long delay times, the feasibility to handle easily the multivariable case and the relative simplicity to deal with constrained control.

Predictive control algorithms are based on an assumed model of the process and on an assumed scene for the future control signals. From the end of 1970's, a lot of predictive control algorithms were presented. Model Algorithmic Control (MAC) (Richalet, Rault, Testud, and Papon 1976, 1978), Dynamic Matrix Control (DMC) (Cutler and Ramaker 1980). Predictive Functional Control (PFC) (Richalet, Abu el Ata-Doss, Arber, Kuntze, Jacubash, and Schill 1987) and Extended Horizon Adaptive Control (EHAC) (Ydstie 1984) are ones of them. The GPC method was proposed by Clarke *et al.* (Clarke, Mohtadi, and Tuffs 1987). It has been studied intensively in the industrial and academic circles and has been successfully implemented in numerous industrial applications (Clarke 1988). It is based on the CARIMA (Controlled Auto-Regressive and Integrated Moving Average) model. The expectation of a quadric function measuring the discrepancy between the predicted output and the predicted reference over prediction horizon plus another quadric function measuring control efforts is the index to be optimized. The optimization results a sequence of control signal. But only the first one is applied to the process to be controlled. A new sequence is calculated during the next sampling interval after a new measurement of output is obtained. This is called receding-horizon control. Based on the same concepts, the ARMA model based GPC is proposed in this paper as an alternative or a complement to enrich the goodness of the GPC.

2. PROCESS MODEL

Assume that the process dynamics are characterized by the local-linearized model

$$x(k) = \frac{B(z^{-1})}{A(z^{-1})}[u(k-d)+v(k-d)] \qquad (1)$$
$$y(k) = x(k)+e(k)$$

where $A(z^{-1})$ and $B(z^{-1})$ are polynomials in the backward shift operator without any common factors, u is the control signal, v is a disturbance acting on the input of the process and e is the measurement noise acting on the process output.

However, when taking into account neither the disturbance nor the measurement noise, the model (1) becomes

$$y(k) = \frac{B(z^{-1})}{A(z^{-1})}u(k-d) \qquad (2)$$

3. OUTPUT PREDICTION

One of the basic ideas of the predictive control is to rewrite the model of the process in order of obtaining an explicit expression for the output at a future time.

3.1. Stable Process

Consider the model (2)

$$y(k) = \frac{B(z^{-1})}{A(z^{-1})} u(k-d) \tag{3}$$

h steps ahead,

$$y(k+h) = \frac{B(z^{-1})}{A(z^{-1})} u(k+h-d) \tag{4}$$

Subtracting Eq. (3) from Eq. (4), we obtain

$$y(k+h) - y(k) = \frac{B(z^{-1})}{A(z^{-1})} u(k+h-d) - \frac{B(z^{-1})}{A(z^{-1})} u(k-d)$$
$$= F_h(z^{-1})u(k+h-d) + \frac{G_h(z^{-1}) - q^{1-d}B(z^{-1})}{A(z^{-1})} u(k-1)$$

where d is the dead time, F_h y $z^{-(h-d+1)}G_h$, whose degree is $h-d$ y $n-1$ respectively, are the quotient and remainder of the division B/A, namely

$$\frac{B(z^{-1})}{A(z^{-1})} = F_h(z^{-1}) + z^{-(h-d+1)} \frac{G_h(z^{-1})}{A(z^{-1})} \tag{5}$$

where

$$F_h(z^{-1}) = f_0 + f_1 z^{-1} + \cdots + f_{h-d} z^{-(h-d)}$$

Define

$$H_h(z^{-1}) = G_h(z^{-1}) - z^{1-d} B(z^{-1}) \tag{6}$$

So

$$y(k+h) = F_h u(k+h-d) + \frac{H_h}{A} u(k-1) + y(k) \tag{7}$$

where the first term of the right hand depends on $u(k), \cdots, u(k+h-d)$, the second term can be treated as a filter and a constant error is assumed over the predication horizon. That is, from Eq. (1) and (7),

$$\varepsilon(k) = \frac{B}{A} v(k) + e(k) \qquad \forall h$$

For simplicity, the last two terms of the right hand side of Eq. (7) are denoted as $\tilde{y}_h(k)$, which is called free response and the first term is called forced response for depending on the future control signals if $h > d$. Thus

$$y(k+h) = F_h(z^{-1})u(k+h-d) + \tilde{y}_h(k) \tag{8}$$

Using the last equation for $h = 1, 2, \cdots, N$, we get

$$\begin{aligned} y(k+1) &= F_1(z^{-1})u(k+1-d) + \tilde{y}_1(k) \\ y(k+2) &= F_2(z^{-1})u(k+2-d) + \tilde{y}_2(k) \\ &\vdots \\ y(k+N) &= F_N(z^{-1})u(k+N-d) + \tilde{y}_N(k) \end{aligned} \tag{9}$$

Introduce vectors

$$\begin{aligned} \mathbf{y} &= [\hat{y}(k+1|k) \quad \hat{y}(k+2|k) \quad \cdots \quad \hat{y}(k+N|k)]^T \\ \mathbf{u} &= [u(k+1|k) \quad u(k+2|k) \quad \cdots \quad u(k+N|k)]^T \\ \tilde{\mathbf{y}} &= [\tilde{y}_1(k) \quad \tilde{y}_2(k) \quad \cdots \quad \tilde{y}_N(k)]^T \end{aligned}$$

where $\hat{y}(k+h|k)$ denotes the h step ahead output prediction made at the instant k and the notation $u(k+h|k)$ denotes the control signal for instant $k+h$ calculated at instant k. Then Eq. (9) can be expressed as

$$\mathbf{y} = \mathbf{F}\mathbf{u} + \tilde{\mathbf{y}} \tag{10}$$

where

$$\mathbf{F} = \begin{bmatrix} f_0 & 0 & \cdots & 0 \\ f_1 & f_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{N-1} & f_{N-2} & \cdots & f_0 \end{bmatrix} \tag{11}$$

is a lower triangular matrix for causality.

As it will be known that the controller based on the predictor (10) incorporate automatically an integrator. In case a double integrator is required in order to follow ramp type reference, the following predictor is needed to reach the purpose.

Multiplying both sides of Eq. (2) with the term $\Delta = 1 - z^{-1}$, which is a difference operator, we get

$$\Delta y(k) = \frac{B(z^{-1})}{A(z^{-1})} \Delta u(k-d)$$

By a similar procedure, we will acquire a predictor for double integral action.

$$\mathbf{y} = \mathbf{F}\Delta\mathbf{u} + \tilde{\mathbf{y}} \tag{12}$$

3.2. Integrating Process

An integrating process can be modeled as

$$y(k) = \frac{B(z^{-1})}{\Delta A(z^{-1})} u(k-d)$$

where d is dead time, $\Delta = 1 - z^{-1}$ is the difference operator as mentioned above and A is stable and well damped. The model can be rewritten as

$$\Delta y(k) = \frac{B(z^{-1})}{A(z^{-1})} u(k-d)$$

From that we can get

$$y(k) = \frac{B(z^{-1})}{A(z^{-1})} u(k-d) - y(k-1) \quad (13)$$

$$y(k+h) = \frac{B(z^{-1})}{A(z^{-1})} u(k+h-d) - y(k+h-1) \quad (14)$$

Subtracting Eq. (13) from Eq. (14), we get

$$y(k+h) = \frac{B}{A} u(k+h-d) - \frac{B}{A} u(k-d) + y(k) - \frac{B(z^{-1})}{A(z^{-1})} u(k-d) + y(k-1)$$

where it is assumed that the sum

$$y(k) - \frac{B(z^{-1})}{A(z^{-1})} u(k-d) + y(k-1)$$

may be different from zero.

Using the identity (5) and the definition (6), iterating the last equation for $h = 1,2,\cdots N$, we get

$$y(k+1) = F_1 u(k+h-d) + \frac{H_1}{A} u(k-1) + (2 - z^{-1}) y(k)$$

$$y(k+2) = F_2 u(k+2-d) + F_1 u(k+1-d) + \frac{H_1}{A} u(k-1) + (3 - 2z^{-1}) y(k)$$

$$\vdots$$

$$y(k+N) = \sum_{i=1}^{N} F_i u(k+i-d) + \sum_{i=1}^{N} \frac{H_i}{A} + (N + 1 - Nz^{-1}) y(k)$$

In vector form

$$\mathbf{y} = \mathbf{Fu} + \mathbf{H} u(k-1) + \mathbf{M} y(k) \quad (15)$$

where the vectors $\mathbf{y}$ and $\mathbf{u}$ are the same as the aforesaid and

$$\mathbf{F} = \begin{bmatrix} F_1 & 0 & \cdots & 0 \\ F_1 & F_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_1 & F_2 & \cdots & F_N \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} \frac{H_1}{A} & \frac{H_1 + H_2}{A} & \cdots & \sum_{i=1}^{N} \frac{H_i}{A} \end{bmatrix}^T$$

$$\mathbf{M} = [2 - z^{-1} \quad 3 - 2z^{-1} \quad \cdots \quad N + 1 - Nz^{-1}]^T$$

Define

$$\tilde{\mathbf{y}} = \mathbf{H} u(k-1) + \mathbf{M} y(k)$$

So Eq. (15) can be written as Eq. (10), and the matrix **F** can be otherwise expressed as

$$\mathbf{F} = \begin{bmatrix} f_0 & 0 & \cdots & 0 \\ f_0 + f_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{N-1} f_i & \sum_{i=0}^{N-2} f_i & \cdots & f_0 \end{bmatrix}$$

where $f_i, i = 0,1,\cdots,N-1$, are the coefficients of the polynomial $F_h(z^{-1})$ as aforementioned.

The controlled system will have double integral action for the process itself has the integrating effect.

3.3. Unstable process

Rewriting the model (2) as

$$A(z^{-1}) y(k) = B(z^{-1}) u(k-d) \quad (16)$$

with A monic and denoting $A(z^{-1})$ as

$$A(z^{-1}) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n} = 1 + A_1 z^{-1}$$

where n is the order of the process, Eq. (16) can thus be rewritten as

$$y(k) = Bu(k-d) - A_1 y(k-1) \quad (17)$$

and h steps ahead

$$y(k+h) = Bu(k+h-d) - A_1 y(k+h-1) \quad (18)$$

Subtracting Eq. (17) from Eq. (18) and assuming the sum $y(k) - Bu(k-d) + A_1 y(k-1)$ could not be zero, we get

$$y(k+h) = Bu(k+h-d) - Bu(k-d) + y(k) + A_1 y(k-1) - A_1 y(k+h-1) \quad (19)$$

Introduce the identity

$$A_1^i B = F_h^i + z^{-(h-d+1)} G_h^i \qquad h = 1,2,\cdots,N \text{ and } i = 0,1,\cdots,h-1$$

where F_h^i contains the first $h-d+1$ terms of $A_1^i B$ and the rest is the second term of the right hand side of the identity. Notice that in the terms F_h^i and G_h^i, i is no more than a notation which indicates the relation between them and $A_1^i B$. The polynomial $[A_1(z^{-1})]^i B(z^{-1})$, $i=0,1,2,\cdots$. is of degree $(i+1)(n-1)$ and expressed in the backward shift operator.

Iterating Eq. (19) for $h=1,2,\cdots,N$, we can get

$$y(k+1)=F_1^0 u(k+1-d)+(G_1^0-z^{1-d}B)u(k-1)+(1-A_1)y(k)+A_1 y(k-1)$$

$$y(k+2)=-F_1^1 u(k+1-d)+F_2^0 u(k+2-d)+[-G_1^1+G_2^0-z^{1-d}B(1-A_1)]u(k-1)+(1-A_1+A_1^2)y(k)+A_1(1-A_1)y(k-1)$$

$$\vdots$$

$$y(k+N)=\sum_{i=1}^{N}F_i^{N-i}u(k+i-d)+\left(\sum_{i=1}^{N}(-1)^{N-i}G_i^{N-i}-z^{1-d}B\sum_{i=1}^{N}(-1)^{i-1}A_1^{i-1}\right)u(k-1)+\sum_{i=1}^{N}(-1)^i A_1^i y(k)+A_1\sum_{i=1}^{N-1}(-1)^i A_1^i y(k-1)$$

The vector form is

$$\mathbf{y}=\mathbf{F}\mathbf{u}+\mathbf{H}u(k-1)+\mathbf{M}y(k) \qquad (20)$$

where the vectors $\mathbf{y}$ and $\mathbf{u}$ are the same as the aforesaid and

$$\mathbf{F}=\begin{bmatrix} F_1^0 & 0 & \cdots & 0 \\ -F_1^1 & F_2^0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^N F_1^N & (-1)^{N-1}F_2^{N-1} & \cdots & F_N^0 \end{bmatrix}$$

$$\mathbf{H}=\begin{bmatrix} G_1^0-z^{1-d}B \\ G_2^0-G_1^1-z^{1-d}B(1-A_1) \\ \vdots \\ \sum_{i=1}^{N}(-1)^{N-i}G_i^{N-i}-z^{1-d}B\sum_{i=1}^{N}(-1)^{i-1}A_1^{i-1} \end{bmatrix}$$

$$\mathbf{M}=\begin{bmatrix} 1-A_1+A_1 z^{-1} \\ 1-A_1+A_1^2+A_1(1-A_1)z^{-1} \\ \vdots \\ \sum_{i=0}^{N}(-1)^i A_1^i+z^{-1}A_1\sum_{i=0}^{N-1}(-1)^i A_1^i \end{bmatrix}$$

For instance, for the first order system

$$y(k)+ay(k-1)=bu(k-1)$$

Following will be resulted

$$\mathbf{F}=b\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -a & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-a)^{N-1} & (-a)^{N-2} & \cdots & 1 \end{bmatrix}$$

$$\mathbf{H}=-b\begin{bmatrix} 1 & 1-a & 1-a+a^2 & \cdots & \dfrac{1-(-a)^N}{1+a} \end{bmatrix}^T$$

$$\mathbf{M}=\begin{bmatrix} 1-a+az^{-1} \\ 1-a+a^2+a(1-a)z^{-1} \\ \vdots \\ \dfrac{1-(-a)^{N+1}+a[1-(-a)^N]z^{-1}}{1+a} \end{bmatrix}$$

By introducing $\tilde{\mathbf{y}}=\mathbf{H}u(k-1)+\mathbf{M}y(k)$, Eq. (20) can be expressed in the same form of Eq. (10).

The predictor developed here can be called general predictor, because it works not only for unstable processes but also for stable processes and integrating ones. An integrating process controlled by a controller based on current predictor owns the double integrating action for the process itself possesses integrating effect.

Through a similar course, we can get a predictor for double integral action.

$$\mathbf{y}=\mathbf{F}\Delta\mathbf{u}+\mathbf{H}\Delta u(k-1)+\mathbf{M}y(k)$$

It can be expressed in the form of (12) if the following is defined.

$$\tilde{\mathbf{y}}=\mathbf{H}\Delta u(k-1)+\mathbf{M}y(k)$$

3.4. Unified predictor

Assume that the polynomial $A(z^{-1})$ with degree *n*, can be factorized as

$$A(z^{-1})=A_2(z^{-1})A_0(z^{-1})$$

where $A_2(z^{-1})$ contains unstable and/or poor damped modes and $A_0(z^{-1})$ contains stable and well damped modes. $A_0(z^{-1})$ and $A_2(z^{-1})$ are monic because so $A(z^{-1})$ is. Therefore $A_2(z^{-1})$ can be expressed as

$$A_2(z^{-1})=1+z^{-1}A_1(z^{-1})$$

Now, the transfer function (2) can be rewritten as

$$y(k)=\frac{B}{A_0}u(k-d)-A_1 y(k-1) \qquad (21)$$

and *h* steps ahead

$$y(k+h)=\frac{B}{A_0}u(k+h-d)-A_1y(k+h-1) \qquad (22)$$

Subtracting Eq. (21) from Eq. (22) and rearranging the terms, we obtain

$$\begin{aligned}y(k+h)&=\frac{B}{A_0}u(k+h-d)-A_1y(k+h-1)\\&+y(k)-\frac{B}{A_0}u(k-d)+A_1y(k-1)\end{aligned} \qquad (23)$$

where it is assumed once again the sum

$$y(k)-\frac{B}{A_0}u(k-d)+A_1y(k-1)$$

may be different from zero, so a constant error is incorporated over the prediction horizon which is, according to the model (1),

$$\varepsilon(k+h)=\frac{B}{A_0}v(k-d)+A_2e(k) \qquad \forall h$$

Let us introduce the identity

$$\frac{A_1^iB}{A_0}=F_h^i+z^{-(h-d+1)}\frac{G_h^i}{A_0}$$

where $h=1,2,\cdots,N$ and $i=1,2,\cdots,N-1$, besides, F_h^i and G_h^i are the quotient and remainder of the division A_1^iB/A_0, respectively. A_1^iB is a polynomial of degree $(i+1)(n-1)$. Through iterating Eq. (23) for all values of h, we get the following.

$$\begin{aligned}y(k+1)&=F_1^0u(k+1-d)+\frac{1}{A_0}(G_1^0-z^{1-d}B)u(k-1)\\&+(1-A_1)y(k)+A_1y(k-1)\\y(k+2)&=-F_1^1u(k+1-d)+F_2^0u(k+2-d)\\&+\frac{1}{A_0}[G_2^0-G_1^1-z^{1-d}B(1-A_1)]u(k-1)\\&+(1-A_1+A_1^2)y(k)+A_1(1-A_1)y(k-1)\\&\vdots\\y(k+N)&=\sum_{i=1}^{N}(-1)^{N-i}F_i^{N-i}u(k+i-d)\\&+\frac{1}{A_0}\left(\sum_{i=1}^{N}(-1)^{N-i}G_i^{N-i}-z^{1-d}B\sum_{i=1}^{N}(-1)^{i-1}A_1^{i-1}\right)u(k-1)\\&+\sum_{i=0}^{N}(-1)^iA_1^iy(k)+A_1\sum_{i=0}^{N-1}(-1)^iA_1^iy(k-1)\end{aligned}$$

The predictions can be expressed in condensed form as

$$\mathbf{y}=\mathbf{Fu}+\mathbf{H}u(k-1)+\mathbf{M}y(k) \qquad (24)$$

where the vectors $\mathbf{y}$ and $\mathbf{u}$ are the same as the aforesaid and

$$\mathbf{F}=\begin{bmatrix}F_1^0 & 0 & \cdots & 0\\ -F_1^1 & F_2^0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ (-1)^NF_1^N & (-1)^{N-1}F_2^{N-1} & \cdots & F_N^0\end{bmatrix} \qquad (25)$$

$$\mathbf{H}=\frac{1}{A_0}\begin{bmatrix}G_1^0-z^{1-d}B\\ G_2^0-G_1^1-z^{1-d}B(1-A_1)\\ \vdots\\ \sum_{i=1}^{N}(-1)^{N-i}G_i^{N-i}-z^{1-d}B\sum_{i=1}^{N}(-1)^{i-1}A_1^{i-1}\end{bmatrix} \qquad (26)$$

$$\mathbf{M}=\begin{bmatrix}1-A_1+A_1z^{-1}\\ 1-A_1+A_1^2+A_1(1-A_1)z^{-1}\\ \vdots\\ \sum_{i=0}^{N}(-1)^iA_1^i+z^{-1}A_1\sum_{i=0}^{N-1}(-1)^iA_1^i\end{bmatrix} \qquad (27)$$

After introducing $\tilde{\mathbf{y}}=\mathbf{H}u(k-1)+\mathbf{M}y(k)$, Eq. (24) can be expressed in the same form of Eq. (10).

Notice that the predictor (24) is identical to that one for stable processes if $A_0=A$, consequently $A_2=1$ and $A_1=0$. The predictor for integrating processes is the case when $A_2=1-z^{-1}$, clearly $A_1=-1$. If $A_2=A$, that means $A_0=1$, the general predictor results.

Analogously, a unified predictor for obtaining increments of control signal can be derived.

$$\mathbf{y}=\mathbf{F}\Delta\mathbf{u}+\mathbf{H}\Delta u(k-1)+\mathbf{M}y(k)$$

It can, of course, be expressed in the form (12) if corresponding definition is introduced.

By introducing

$$A_I=\begin{cases}A_1 & \text{for obtaining } u(t)\\ A_1-A_2 & \text{for obtaining } \Delta u(t)\end{cases}$$

a general expression of identity can be found

$$\frac{A_I^iB}{A_0}=F_h^i+z^{-(h-d+1)}\frac{G_h^i}{A_0} \qquad \forall h \text{ and } \forall i \qquad (28)$$

For the last, if introduce the following notation:

$$\mathbf{w} = \begin{cases} \mathbf{u} & \text{for obtaining } u(t) \\ \Delta\mathbf{u} & \text{for obtaining } \Delta u(t) \end{cases} \quad (29)$$

all the above-mentioned predictors can be expressed in a unique form

$$\mathbf{y} = \mathbf{F}\mathbf{w} + \tilde{\mathbf{y}} \quad (30)$$

4. OBJECTIVE FUNCTION

Different objective functions have been proposed for different model predictive control (MPC) algorithms. But the aim is basically the same. The distance between the future output and the reference and the control efforts are penalized. The general expression for such an objective function is

$$J(N_1, N_2, N_u) = \sum_{i=N_1}^{N_2} \lambda(i)[y(k+i) - r(k+i)]^2 + \sum_{j=1}^{N_u} \rho(j)[\Delta^2 u(k+j-1)]^2 \quad (31)$$

where N_1 and N_2 are the minimum and maximum prediction horizons and N_u is the control horizon. $\lambda(i)$ and $\rho(j)$ are coefficients whose elections depend on pretension to future behavior of the system. The signal r is the reference.

One of the advantages of predictive control is that the system can react before the change has effectively been made if the future evolution of the reference is known a priori. That, consequently, avoids the effect of delay in the process response.

It is needed to redefine the objective function in order to obtain an optimal sequence of increments of control signal.

$$J(N_1, N_2, N_u) = \sum_{i=N_1}^{N_2} \lambda(i)[y(k+i) - r(k+i)]^2 + \sum_{j=1}^{N_u} \rho(j)[\Delta^2 u(k+j-1)]^2 \quad (32)$$

A reference trajectory is frequently used in the objective functions (31) and (32):

$$y_m(k) = y(k)$$
$$y_m(k+h) = \alpha y_m(k+h-1) + (1-\alpha) r(k+h) \qquad h = 1, \cdots, N$$

where α is a parameter of design and takes the value between zero and one, the closer to unity the smoother the approximation.

In this paper $\rho(j) = \text{constant}, \forall j$ and $\lambda(i) = 1, \forall i$, are adopted for the sake of simplicity, unless otherwise indicated.

5. CONTROL LAW

The aim of the MPC is that the future output on considered horizon should follow a determined reference signal but the control effort is taken into account too. Thus the objective function should be minimized. The product of the minimization is the control law. Namely, the objective function will be minimized by using the resultant control law. For this purpose, the following are introduced.

$$\mathbf{r} = [r(k+1) \quad r(k+2) \quad \cdots \quad r(k+N)]^T$$
$$\mathbf{u}(k) = [u(k+1-d \mid k) \quad \cdots \quad u(k+N-d \mid k)]^T$$
$$\mathbf{u}(k-1) = [u(k+1-d \mid k-1) \quad \cdots \quad u(k+N-d \mid k-1)]^T$$

hence

$$\Delta\mathbf{u} = [\mathbf{u}(k) - \mathbf{u}(k-1)]^T$$

Assume $N_1 = 1$ without any loss of generality. The cost function (31) can be written as

$$J(1, N_2, N_u) = (\mathbf{y} - \mathbf{r})^T (\mathbf{y} - \mathbf{r}) + \rho \Delta\mathbf{u}^{\mathbf{T}} \Delta\mathbf{u}$$
$$= [\mathbf{F}\mathbf{u}(k) + \tilde{\mathbf{y}} - \mathbf{r}]^T [\mathbf{F}\mathbf{u}(k) + \tilde{\mathbf{y}} - \mathbf{r}] + \rho[\mathbf{u}(k) - \mathbf{u}(k-1)]^T [\mathbf{u}(k) - \mathbf{u}(k-1)]$$

where Eq. (10) is used as a general case. Minimizing the cost function with respect to $\mathbf{u}(k)$, we get

$$\mathbf{u}(k) = (\mathbf{F}^T \mathbf{F} + \rho \mathbf{I})^{-1} [\mathbf{F}^T (\mathbf{r} - \tilde{\mathbf{y}}) + \rho \mathbf{u}(k-1)] \quad (33)$$

The first element $u(k)$ of vector $\mathbf{u}(k)$ is applied to the process. The control law is calculated again when a new measurement is obtained at the next sampling instant. Thus, the receding-horizon control concept is used.

The control law (33) incorporates implicitly the integral action. This will be proved below. It is time invariant if the process to be controlled is time invariant.

A matrix of dimension $N \times N$ has to be inverted in the calculation of the control law, where N is the prediction horizon. It is possible to introduce constraints on the future control signal in order to decrease the computations. Assume, for example, that the control signals are constant after N_u steps, with $N_u < N$,

$$u(k+N_u) = u(k+N_u+1) = \cdots = u(k+N)$$

This implies that the control increments are assumed to be zeros after N_u steps. The control law (33) should be modified to

$$\mathbf{u}(k) = (\mathbf{F}_1^T \mathbf{F}_1 + \rho \mathbf{I})^{-1} [\mathbf{F}_1^T (\mathbf{r} - \tilde{\mathbf{y}}) + \rho \mathbf{u}(k-1)]$$

where $\mathbf{F}_1$ is a $N \times N_u$ matrix. Therefore, the matrix to be inverted is now $N_u \times N_u$ dimensions. For instance, applying mentioned assumption, the matrix described in Eq. (11) will be

$$\mathbf{F_1} = \begin{bmatrix} f_0 & 0 & \cdots & 0 & 0 \\ f_1 & f_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{N_u-1} & f_{N_u-2} & \cdots & f_1 & f_0 \\ f_{N_u} & f_{N_u-1} & \cdots & f_2 & f_1 + f_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{N-1} & f_{N-2} & \cdots & f_{N-N_u+1} & \sum_{i=0}^{N-N_u} f_i \end{bmatrix}$$

By an analogous procedure and with the predictor (12), we can get an optimal sequence of increments of the control signal for double integral action.

$$\Delta\mathbf{u}(k) = (\mathbf{F}^T\mathbf{F} + \rho\mathbf{I})^{-1}[\mathbf{F}^T(\mathbf{r} - \tilde{\mathbf{y}}) + \rho\Delta\mathbf{u}(k-1)] \quad (34)$$

The control law can be expressed as

$$\begin{aligned} \Delta u(k) &= \mathbf{L}[\mathbf{F}^T(\mathbf{r} - \tilde{\mathbf{y}}) + \rho\Delta\mathbf{u}(k-1)] \\ u(k) &= u(k-1) + \Delta u(k) \end{aligned} \quad (35)$$

where $\mathbf{L}$ is the first row of the matrix $(\mathbf{F}^T\mathbf{F} + \rho\mathbf{I})^{-1}$.

Now let us demonstrate one of the properties of the controller (33). Firstly we introduce a definition.

DEFINITION 1 Estimation of Output

The vector given by $\hat{\mathbf{y}} = \mathbf{F}\mathbf{u}(k-1) + \tilde{\mathbf{y}}$ is a estimation of the vector given by $\mathbf{y} = \mathbf{F}\mathbf{u}(k) + \tilde{\mathbf{y}}$ with the available data at instant k before calculating $\mathbf{u}(k)$, because $\mathbf{u}(k-1)$ is obtained before instant k and $\tilde{\mathbf{y}}$ is obtained at instant k.

THEOREM 1 Integral Action

The control law (33) derived by means of minimizing the cost function (31) possesses the integral action.

Proof: Assume $N_1 = d = 1$ without any loss of generality. The control law can be rewritten as

$$(\mathbf{F}^T\mathbf{F} + \rho\mathbf{I})\mathbf{u}(k) - \rho\mathbf{u}(k-1) = \mathbf{F}^T(\mathbf{r} - \tilde{\mathbf{y}})$$

Subtracting the term $\mathbf{F}^T\mathbf{F}\mathbf{u}(k-1)$ on both sides of the last equation, using the definition and gathering the terms, we get

$$(\mathbf{F}^T\mathbf{F} + \rho\mathbf{I})[\mathbf{u}(k) - \mathbf{u}(k-1)] = \mathbf{F}^T(\mathbf{r} - \hat{\mathbf{y}})$$

Thus

$$\mathbf{u}(k) = \mathbf{u}(k-1) + (\mathbf{F}^T\mathbf{F} + \rho\mathbf{I})^{-1}\mathbf{F}^T(\mathbf{r} - \hat{\mathbf{y}})$$

In words, the control law (33) has the integral action. □

It deserves to be mentioned that a controlled system will have offset in steady-state if u instead of Δu is penalized. But it is an alternative penalizing u when a regulator is wanted.

Now let us introduce the suboptimal concept. When replacing the vector $\mathbf{u}(k-1) = [u_h(k-1)]$, $h = 1, \cdots, N$, with $\mathbf{1}u_1(k-1)$, where $\mathbf{1} = [1 \ \ 1 \ \ \cdots \ \ 1]^T$ an $N \times 1$ vector, in the calculating of the control signal, a suboptimal controller is obtained. The suboptimal concept can be used to reduce computation burden and overshoot. For example, the process $G(s) = 1/(s+1)$ is controlled by a controller with following parameters $N = 8, N_u = 1$ and $\rho = 0.1$. Its step response is drawn in Fig.1 (blue trajectory). Assume a quicker recovery from disturbance is required. It can be reached with larger N_u. The red trajectory (left drawing of Fig.1) is resulted when $N_u = 5$. Larger N_u means larger burden of computation. But it can be reduced by using the suboptimal concept as above-mentioned. The result is plotted in the left drawing of Fig. 1 (dashed). Notice that the dashed one and the red one are very similar. Here the reference trajectory which was mentioned in Section IV is used to lower the overshoots. However, the suboptimal concept can also be used to change other dynamic aspects of a controlled system. Fig.1 (right) depicted the result of using it to cut down the overshoot of the same process given above. After applying the suboptimal concept, the dashed trajectory is gotten.

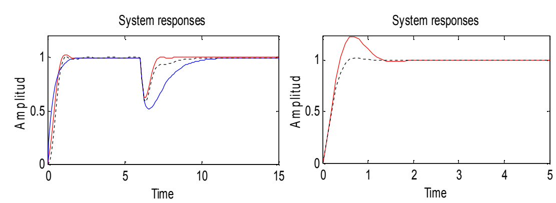

Figure1. Step Responses of the System. The Left: $N_u = 1$ (blue), $N_u = 5$ (red) and $N_u = 2$ (Dashed). A Load Disturbance Is Introduced at Time 6. The Right: Results of Applying the Suboptimal Concept to Reduce Overshoot.

In order to demonstrate the double integral action of the control law (35), the following definition is introduced.

DEFINITION 2 Estimation of Output

The vector given by $\hat{\mathbf{y}} = \mathbf{F}\Delta\mathbf{u}(k-1) + \tilde{\mathbf{y}}$ is a estimation of the vector given by $\mathbf{y} = \mathbf{F}\Delta\mathbf{u}(k) + \tilde{\mathbf{y}}$ with the available data at instant k, because $\mathbf{u}(k-1)$ is obtained before instant k and $\tilde{\mathbf{y}}$ is obtained at instant k.

THEOREM 2 Double Integral Action

The control law (35) derived by means of minimizing the cost function (32) possesses the double integral action.

Proof: Assume $N_1 = d = 1$ without any loss of generality. Eq. (34) can be rewritten as

$$(\mathbf{F}^T\mathbf{F}+\rho\mathbf{I})\Delta\mathbf{u}(k)-\rho\Delta\mathbf{u}(k-1)=\mathbf{F}^T(\mathbf{r}-\tilde{\mathbf{y}})$$

Subtracting the term $\mathbf{F}^T\mathbf{F}\Delta\mathbf{u}(k-1)$ *on both sides of the last equation and manipulating algebraically, we get*

$$\Delta\mathbf{u}(k)=\Delta\mathbf{u}(k-1)+(\mathbf{F}^T\mathbf{F}+\rho\mathbf{I})^{-1}\mathbf{F}^T(\mathbf{r}-\hat{\mathbf{y}})$$

where $\hat{\mathbf{y}}=\mathbf{F}\Delta\mathbf{u}(k-1)+\tilde{\mathbf{y}}$ *as defined. Thus the sequence* $\Delta\mathbf{u}$ *is obtained by means of integrating. That signifies the increment* $\Delta u(k)$ *is given by integration. And another integral action is given by*

$$u(k)=u(k-1)+\Delta u(k)$$

Therefore the double integral action of the control law (35) is revealed. □

Previous theorems are illustrated with some simple examples. All of them are the particular case with following assumptions: constant future control, namely $u(k+i)=u(k),\quad i=1,2,\cdots,h,$ there is no constraint on the control effort and it is desired that $y(k+h)=r(k+h)$, where h is the prediction horizon. In other words this is the case when $N_u=1$, $\rho=0$ and $\lambda(i)=0,\ i=1,2,\cdots,h-1$ but $\lambda(h)=1$.

Example 1. Consider a first order process

$$y(k)+ay(k-1)=bu(k-1)$$

If it is open-loop stable and well damped, we can use the predictor (10) for stable process case. After algebraic manipulations, we obtain the control law

$$u(k)=u(k-1)+\frac{(1+a)(1+az^{-1})}{b[1-(-a)^h]}[r(k+h)-y(k)]$$

We can also use the predictor for unstable process case to obtain the control law no matter whether the process may be stable or not.

$$u(k)=u(k-1)+\frac{1+a}{b[1-(-a)^h]}\left\{r(k+h)-\left[\alpha+\beta z^{-1}\right]y(k)\right\}$$

where

$$\alpha=\frac{a[1-(-a)^{h+1}]}{1+a}\qquad\qquad \beta=\frac{a[1-(-a)^h]}{1+a}$$

Example 2. Consider the same process of Example 1 and assume it is open-loop stable and well damped. It is desired that the output of the process can follow a mixed trajectory composed of steps and ramps. So the double integral action is needed. Using the predictor (12) and manipulating algebraically, we will finally get

$$\Delta u(k)=\Delta u(k-1)+\xi[r(k+h)-(h+1-hz^{-1})y(k)]$$
$$u(k)=u(k-1)+\Delta u(k)$$

where

$$\xi=\frac{(1+a)^2(1+az^{-1})}{b[h(1+a)+a+(-a)^{h+1}]}$$

Example 3. Consider an integrating process

$$y(k)=\frac{b_1+b_2z^{-1}}{(1-z^{-1})(1+az^{-1})}u(k-1)$$

with the pole $p=-a$ in the unit disc and well damped. Using the corresponding predictor, we can get the following control law through same manipulations.

$$u(k)=u(k-1)+\eta[r(k+h)-(h+1-hz^{-1})y(k)]$$

where $\eta=\dfrac{(1+a)(1+az^{-1})}{h(b_1+b_2)-\dfrac{b_2-b_1a}{1+a}[1-(-a)^h]+h(1+a)b_2z^{-1}}$

6. STABILITY AND ROBUSNESS

No Mathematic model is able to describe exactly a physical process. Some approximations are always made. However, it is desired that the controlled system should be insensitive to those uncertainties in the model.

In the absence of theoretical results, some numerical analyses are presented.

Let $N_1=d$ and assume the transport delays are multiple of the sampling time for all cases. Only a few types of uncertainties are considered here.

Case 1. Consider a stable process

$$G(s)=\frac{b}{(s+a)}e^{-\tau s}$$

with nominal values $a=2$ and $b=0.5$, which is sampled with sampling time 0.1 seconds, so the discrete nominal plant is

$$y(k)-0.8187y(k-1)=0.0453u(k-d)$$

The nominal system is stable for all $N\geq d$, and $1\leq N_u\leq N-d+1$.

The simulations with $N-N_1=0, N_u=1$ and $\rho=0.1$ show the followings.

1. Uncertainty at the pole: When $d=1$, the system is stable for $a\geq 0.01$. That is to say an uncertainty of about ±99.5% is permitted.

When $d=10$, the system keeps stable for $a \geq 0.6$. That means an uncertainty of about $\pm 70\%$ is permitted. Fig. 2 is step responses and Nyquist diagrams.

2. Uncertainty of the gain: When $d=1$, the system dose not loss stability for $0.01 \leq b \leq 80$. Namely a variation about $\pm 98\%$ is allowed. When $d=10$, the system is stable for $0.01 \leq b \leq 1.6$. That is a variation about $\pm 98\%$ is allowed. See Fig. 2.
3. Unmodelled pole: Assume that the real process has another pole -2σ but its static gain maintains the same in spite of the existence of another pole. When $d=1$, the system is stable for $\sigma \geq 0.1$. When $d=10$, the system is also stable for $\sigma \geq 0.1$. Fig. 3 shows step responses of the controlled system and Nyquist diagrams when the real process has a less dominant pole -10.
4. Uncertainty at the delay: When $d=1$, an error of 12 units is tolerated. When $d=10$, an error of 19 units is tolerated and for all the possible negative values of delay mismatch, from -1 to -9, the controlled system does not lose stability. See Fig. 4.

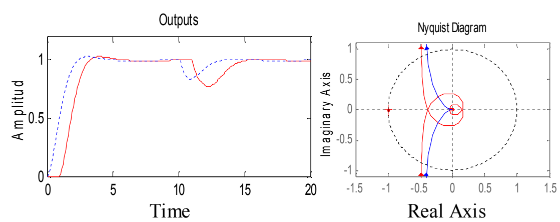

Figure 2. Responses of the Nominal System to Step Type Reference and Load Disturbance (Left), and Nyquist Diagrams (Right) when $d=1$ (Blue) and $d=10$ (Red).

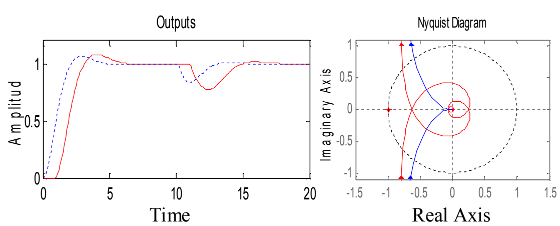

Figure 3. Responses of the System to Step Type Reference and Load Disturbance (Left), and Nyquist Diagrams (Right) with Unmodelled Mode when $d=1$ (Blue Trajectory) and $d=10$ (Red Trajectory).

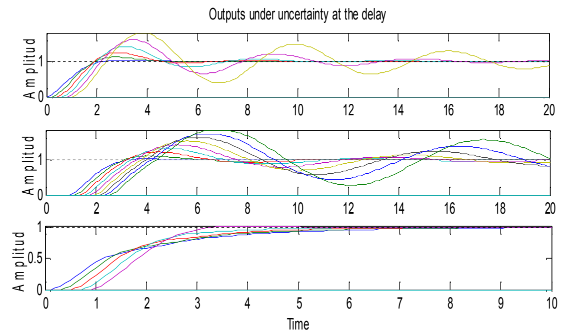

Figure 4. Step Responses of the Controlled System under the Uncertainty at the Delay, when $d=1$ (Above) and $d=10$ (Middle and Below). Only Those with Even Numbers of Delay Units Are Depicted for Clarity.

Case 2. Consider a stable second order process

$$G(s)=\frac{\omega^2}{s^2+2\zeta\omega s+\omega^2}e^{-\tau s}$$

with nominal values $\zeta=0.5$ and $\omega=1$ rad/s, which means the process has poles at $-0.5 \pm j0.8660$. It is sampled with sampling time 0.1 seconds.

$$y(k)=\frac{0.0047+0.0044z^{-1}}{1-1.8097z^{1}+0.8187z^{-2}}u(k-d)$$

The nominal system is stable for $N-N_1=0$ and $N-N_1 \geq 4$. The simulations with $N-N_1=7, N_u=1$ and $\rho=0.1$ obtain the following results.

1. Uncertainty of the damping: When $d=1$, the system is stable for $0.01 \leq \zeta \leq 10$. That is to say an uncertainty of about $\pm 98\%$ is permitted. When $d=10$, the system keeps stable for $0.2 \leq \zeta \leq 10$. That means an uncertainty of about $\pm 30\%$ is permitted. Fig. 5 is the step responses of the controlled nominal process and the correspondent Nyquist diagrams.
2. Uncertainty of the nature frequency: When $d=1$, the system does not lose stability for $0.45 \leq \omega \leq 2.5$. Namely a variation about $\pm 55\%$ is allowed. When $d=10$, the system is stable for $0.5 \leq \omega \leq 1.35$. That is a variation about $\pm 35\%$ is allowed. See Fig. 5.
3. Unmodelled pole: Assume that the real process has another pole -0.5σ but its static gain maintains the same in spite of the existence of another pole. When $d=1$, the system is stable for $\sigma \geq 0.3$. When $d=10$, the system is also stable for $\sigma \geq 0.3$. Fig. 6 shows the step responses and the Nyquist diagrams when the

real process has a less dominant pole -10. If the real process has a couple of conjugated complex poles $-0.5\sigma \pm j0.8660\varpi$ is assumed, the controlled system is stable, for example, for $\sigma \geq 3, \varpi = 1$ or for $\sigma = 1, \varpi \geq 5$ when $d = 1$, and for $\sigma \geq 1.5, \varpi = 1$ or $\sigma = 1, \varpi \geq 2$ when $d = 10$. A drawing similar to Fig. 6 can be obtained under the assumption that the real process has another less dominant couple of poles $-10 \pm j8.660$.

4. Uncertainty at the delay: When $d = 1$, an error of 6 units is tolerated. When $d = 10$, an error of 14 units is tolerated and for all the possible negative values of delay mismatch, from -1 to -9, the controlled system remains stable. See Fig. 7.

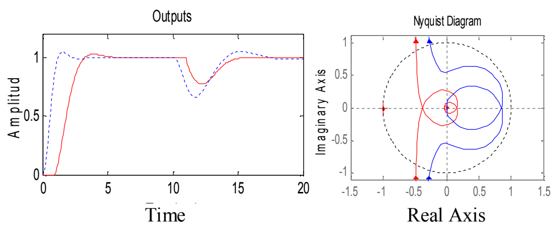

Figure 5. Responses of Nominal System to Step Type Reference and Load Disturbance (Left), and Nyquist Diagrams (Right) when $d = 1$ (Blue) and $d = 10$ (Red).

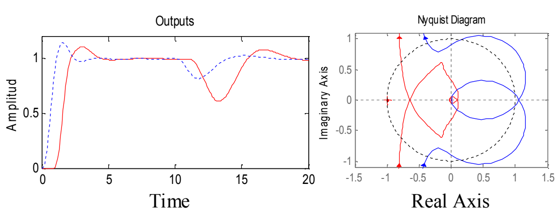

Figure 6. System's Responses to Step Type Reference and Load Disturbance (Left), and Nyquist Diagrams (Right) of the System with Unmodelled Mode when $d = 1$ (Blue) and $d = 10$ (Red).

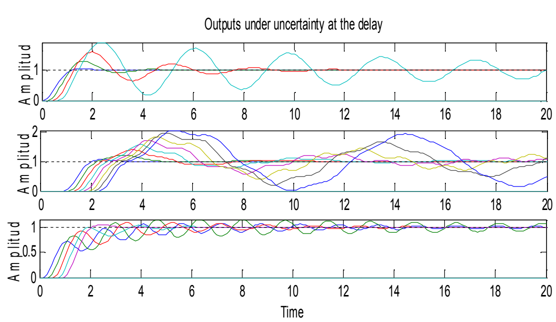

Figure 7. Responses of the Controlled System to Step Type Reference under the Uncertainty at the Delay, when $d = 1$ (Above) and $d = 10$ (Middle and Below). Only Those with Even Numbers of Delay Units Are Depicted for Clarity.

Case 3. Consider a second order integrating process

$$G(s) = \frac{a}{s(s+a)} e^{-\tau s}$$

with nominal values $a = 2$. It is sampled with sampling time 0.1 seconds.

$$y(k) = \frac{0.0094 + 0.0088z^{-1}}{(1-z^{1})(1-0.8187)z^{-2}} u(k-d)$$

The nominal system is stable for $N \geq 4$ and $N_u = 1$ when $d = 1$. It is stable for $N \geq 10$ and $N_u = 1$ when $d = 10$. For this particular case, the controlled system is not always stable for $N_u \leq N - d + 1$. The simulations with $N - N_1 = 7, N_u = 1$ and $\rho = 0.1$ when $d = 1$ and with $N - N_1 = 32, N_u = 1$ and $\rho = 0.1$ when $d = 10$ show the followings.

1. Uncertainty of the parameter a: When $d = 1$, the system is stable for $0.01 \leq a \leq 20$. That is to say an uncertainty of about $\pm 98\%$ is permitted. When $d = 10$, the system maintains stability for $0.9 \leq a \leq 20$. That means an uncertainty of about $\pm 55\%$ is permitted. Fig. 8 is the step responses of the controlled nominal process and the correspondent Nyquist diagrams.
2. Unmodelled pole: Assume that the real process has another pole -2σ but its static gain maintains the same in spite of the existence of another pole. When $d = 1$, the system is stable for $\sigma \geq 2.4$. When $d = 10$, the system is stable for $\sigma \geq 3.5$. Fig. 9 shows the step responses and the Nyquist diagrams when the real process has a less dominant pole -10 or -20.

4. Delay uncertainty: When $d = 1$, an error of 2 units is tolerated and up to 4 is tolerated when $N = 18$. When $d = 10$, an error of 6 units is tolerated and up to 10 units is tolerated when $N = 50$. And for all the possible negative values of delay mismatch, from -1 to -9, the controlled system preserves stability. See Fig. 10. It is noticed that the tolerance is dependent on prediction horizon.

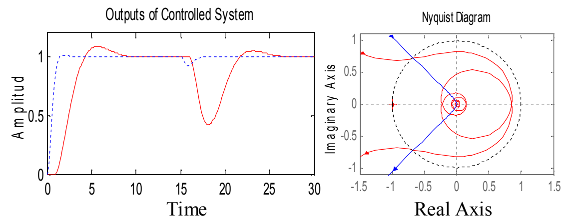

Figure 8. Responses of the Nominal System to Step Type Reference and Load Disturbance (Left), and Nyquist Diagrams (Right) when $d = 1$ (Blue Trajectory) and $d = 10$ (Red Trajectory).

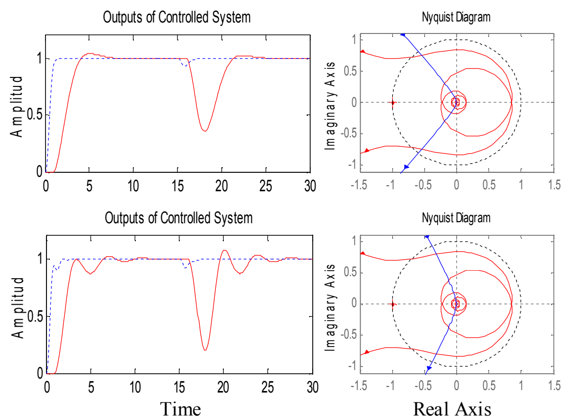

Figure 9. Responses of the Controlled System to Step Type Reference and Load Disturbance (Left), and Nyquist Diagrams (Right) with Unmodelled Mode when $d = 1$ (Blue Trajectory) and $d = 10$ (Red Trajectory) and when $\sigma = 10$ (Superior) and $\sigma = 5$ (Inferior).

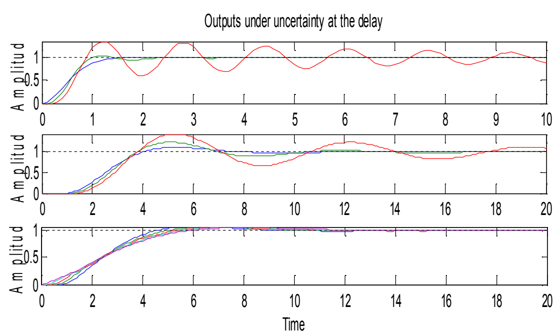

Figure 10. Step Responses of the System under the Uncertainty at the Delay, when $d = 1$ (Above) and $d = 10$ (Middle and Below, Only Those with Odd Numbers of Delay Units Are Depicted for Clarity).

Three numerical analyses of controlled systems have been given. Those controllers were based on the predictors for stable processes or for integrating processes. In the following the controller based on the predictor for unstable processes will be used in order to compare the proposed controller with that based on the CARIMA model.

The CARIMA model is

$$A(z^{-1})y(k) = z^{-d}B(z^{-1})u(k) + C(z^{-1})\frac{e(k)}{\Delta}$$

where e is a white noise. The comparisons will de made under the conditions $C(z^{-1}) = 1$ for the CARIMA model, and $S_1(z^{-1}) = 1, T_1(z^{-1}) = 1$ (see Section VII for the definitions) for the ARMA model.

The process to be controlled is a stable one.

$$y(k) - ay(k-1) = bu(1-d)$$

For the range of variation of the parameters: $0.1 \le a \le 0.98$ and the possible values of gain with which the controlled system keeps stable. The numerical results show that no difference, in respect of tracking a reference and rejecting disturbances, can be observed between the controller based on the predictor for unstable processes and that based on CARIMA model when $d = 1$. However, when $d = 10$, the pole is big $a = 0.98$, and the process gain is 5, the system controlled by controller based on predictor for unstable processes can tolerate a delay estimation error of 7 units. See Fig. 10. Nevertheless, for a delay of 10 units a delay mismatch of one unit is permitted by the controller based on CARIMA model (Camacho, Bordons, 2007).

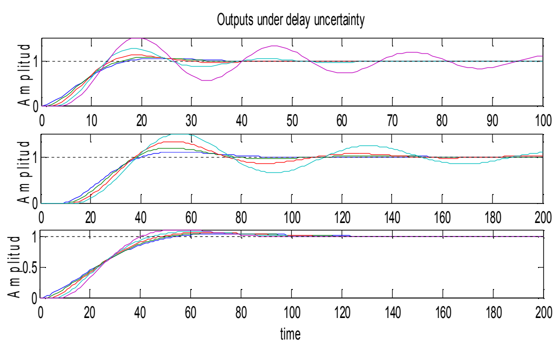

Figure11. Step Responses of the System under Delay Uncertainties, when $d = 1$ (Above) and $d = 10$ (Middle and Below, Only Those with Even Numbers of Delay Units Are Depicted for Clarity).

7. A CLASSICAL POINT OF VIEW

The terms T and R can be incorporated when the controller (33) or (35) is viewed as a classical controller,

$$R(z^{-1})u(k) = T(z^{-1})r(k) - S(z^{-1})y(k) \qquad (36)$$

For doing so, introduce

$$y_f(k) = \frac{S_1(z^{-1})}{T_1(z^{-1})}y(k) \qquad (37)$$

with $S_1(1)/T_1(1)=1$ for avoiding static error, and factorize A as $A=A_2A_0$ and use the identity (28) in order to get a general sense result. Thereby the model (2) can be rewritten as

$$A_2(z^{-1})y_f(k)=\frac{S_1(z^{-1})}{A_0(z^{-1})T_1(z^{-1})}w(k-d)$$

where w is defined as (29).

By a similar way to that in Section II, we can get a predictor with the form of Eq. (30). It can be also expressed in the form of Eq. (19),

$$y_f=\mathbf{F}\mathbf{u}+\mathbf{H}u(k-1)+\mathbf{M}y_f(k) \tag{38}$$

where $\mathbf{F}$ has the same form of (25), replacing the polynomials T_1, B and A_1 of the Eq. (26) and (27) with A_0T_1, BS_1 and A_I, respectively, we can get the new matrixes $\mathbf{H}$ and $\mathbf{M}$ for Eq. (38).

Let $\Gamma=[\gamma_i], i=1,2,\cdots,N$ be the first row of the matrix $(\mathbf{F}^T\mathbf{F}+\rho\mathbf{I})^{-1}\mathbf{F}^T$ and $\mathbf{1}=[1\ \ 1\ \ \cdots\ \ 1]^T$ an $N\times 1$ vector, where N is the prediction horizon.

Eq. (33) and (34) can be expressed as

$$\mathbf{w}(k)=(\mathbf{F}^T\mathbf{F}+\rho\mathbf{I})^{-1}[\mathbf{F}^T(\mathbf{r}-\tilde{\mathbf{y}})+\rho\mathbf{w}(k-1)]$$

It is equivalent to

$$(\mathbf{F}^T\mathbf{F}+\rho\mathbf{I})\mathbf{w}(k)-\rho\mathbf{1}w(k-1)=\mathbf{F}^T(\mathbf{r}-\tilde{\mathbf{y}})$$

because the receding-horizon concept is used. Subtracting $\mathbf{F}^T\mathbf{F}\mathbf{1}w(k-1)$ from both sides of the last equation and manipulating algebraically, we get

$$\mathbf{w}(k)-\mathbf{1}w(k-1)=(\mathbf{F}^T\mathbf{F}+\rho\mathbf{I})^{-1}\mathbf{F}^T[\mathbf{r}-\tilde{\mathbf{y}}-\mathbf{F}\mathbf{1}w(k-1)]$$

However, only the first element of the vector is used, therefore the following results.

$$(1-z^{-1})w(k)=\Gamma[\mathbf{r}-\tilde{\mathbf{y}}-\mathbf{F}\mathbf{1}w(k-1)]$$

Assume $N_1=1$ without any loss of generality and consider that the future reference keeps constant along the horizon or its evolution is unknown. That is $w(k+h)=w(k), \forall h$. By means of algebraic manipulations, we can get

$$\begin{aligned}(1-z^{-1})R_1w(k)&=A_0T_1\sum_{i=1}^{N}\gamma_i r(k)\\&-A_0S_1\sum_{i=1}^{N}\gamma_i\sum_{j=1}^{i}[1+(-1)^jA_I^j(1-z^{-1})]y(k)\end{aligned}$$

where

$$\begin{aligned}(1-z^{-1})R_1&=(1-z^{-1})A_0T_1\\&+z^{-1}\sum_{i=1}^{N}\gamma_i\sum_{j=1}^{i}(-1)^{i-j}\left[(1-z^{-i})G_j^{i-j}+(1-z^{1-d})A_I^{i-j}BS_1\right]\\&=(1-z^{-1})A_0T_1\\&+(1-z^{-1})z^{-1}\sum_{i=1}^{N}\gamma_i\sum_{j=1}^{i}(-1)^{i-j}\left[P_1^{1-i}G_j^{i-j}+P_2^{2-d}A_I^{i-j}BS_1\right]\end{aligned}$$

where P^l denotes a polynomial of degree l. The second equality is obtained for the fact that a polynomial $1-z^{-m}$, m integers, has always a factor $1-z^{-1}$ is used (De Moivre's theorem). Thus compare the third equation from the bottom with Eq. (36), we get

$$\begin{aligned}R&=(1-z^{-1})R_1\\S&=A_0S_1\sum_{i=1}^{N}\gamma_i\sum_{j=1}^{i}[1+(-1)^jA_I^j(1-z^{-1})]\\T&=A_0T_1\sum_{i=1}^{N}\gamma_i\end{aligned}$$

The controlled system is, according to the model (1),

$$\begin{aligned}A_2x(k)&=\frac{z^{-d}B}{A_0}[u(k)+v(k)]\\y(k)&=x(k)+e(k)\\Ru(k)&=Tr(k)-Sy(k)\end{aligned} \tag{39}$$

where $A_0A_2=A$ as foresaid. By resolving Eq. (39) for x, y and u, we will get the characteristic equation.

$$AR+z^{-d}BS=0$$

A proper selection of the term or filter T_1 can improve the robustness. Signals with certain frequencies can be suppressed by requiring that the polynomial $S(z^{-1})$ vanishes at corresponding values of z. There is no steady-state error when the load disturbance is a step for the existence of an integrator. That is $R(1)=0$. Due to the latter, the present section can be regarded as another proof of Theorem 1.

Notice that the obtained controller is dependent on prediction horizon and on delay as well if $d>1$. It is not obvious whether there is virtue for this distinguishing feature. Simulations show that a small N is enough to stabilize a system and that the sensitivity of a controlled system is closely related to prediction horizon.

8. CONCLUSION

According to simulations, the ARMA model based GPC presented here can deal with a great variety of processes, stable and unstable ones as well as those of nonminimum phase. The predictor formulated for unstable process can be also used for stable ones as well as for integrating ones. However, there is difference with respect to the dynamic response of the controlled system to disturbances and to uncertainties when different predictor is used. To say roughly, a system controlled by a controller based on the predictor for unstable processes is less robust but more rapid to recover from a disturbance. The derived relationship between GPC and RST controllers indicates how to improve the controller if it is needed. There are potential applications of proposed controllers in the case of poor estimation of process delays. All the matrixes involved $\mathbf{F}$, $\mathbf{H}$ and $\mathbf{M}$, can be calculated recursively. Present method presents difficulties when it is used to control unstable and simultaneously nonminimum phase processes. Some of them can be controlled with large N and $N_u \geq 2$.

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