ROBUST TRAJECTORY TRACKING PROBLEM: CHEAP CONTROL APPROACH

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ABSTRACT
A robust trajectory tracking problem is treated in the framework of a zero-sum linear-quadratic differential game of a general type. For a cheap control version of this game, a novel solvability condition is derived. Condition, guaranteeing that the tracking problem is solved by a cheap control game optimal strategy, is established. A boundedness of the minimizer's control is analyzed. Illustrative pursuit-evasion examples are presented.

Keywords: Trajectory tracking, robust control, linear-quadratic differential game, cheap control

1. INTRODUCTION

The problem of tracking a given trajectory under uncertainties (trajectory planning, path following etc.) is a well-known challenge in aerospace (Ben-Asher et al., 2004; Zhang et al., 2008), underwater vehicles control (Aguiar & Hespanha, 2007; Kiselev, 2009), robotics (Wang et al., 2009; Kowalczyk et al., 2009)and many other applications. Most of the approaches, known in the literature, provide the trajectory tracking asymptotically for time tending to infinity (see e.g. (Aguiar & Hespanha, 2007; Sun et al., 2009; Cheng et al., 2007)). In many practical applications, for example, in aerospace, the tracking should be guaranteed on a finite time interval. In real-life systems, the trajectory should be tracked in the presence of uncertainty and/or disturbance. However, to the best knowledge of the authors, only a small number of the papers considers the tracking problem from this viewpoint (see e.g. (Pei et al., 2003; Mahony & Hamel, 2004)). In (Basar & Bernhard, 1995; Tretyakov & Turetsky, 1995; Ben-Asher et al., 2004), the tracking problem is formulated on a finite time horizon in the framework of a differential game. In this paper, based on such a formulation, the tracking problem is solved by using a cheap control linear-quadratic approach.

Condition for solvability of the linear-quadratic differential game (LQDG) was first formulated by Bernhard (1979; 1980) as lack of conjugate points on a game time interval. This condition, although being necessary and sufficient, cannot be verified directly. Thus, for the LQDG, the conditions, guaranteeing the lack of the conjugate points, are of a great importance. A number of works, dealing with this issue, can be mentioned. In (Basar & Bernhard, 1995), it was established that the game solution exists if the maximizer's control cost in the performance index is sufficiently large. In (Reid, 1972), the game solvability follows from the invertibility of the solution of some matrix linear differential equation. Due to (Mou & Liberty, 2001), the game solvability is provided by the existence of so-called lower and upper solutions of the matrix Riccati differential equation, associated with the game. These conditions do not provide a direct verification scheme based on the dynamics and cost functional coefficients. This drawback was partially surmounted by the condition, formulated in (Shinar *et al.*, 2008) in terms of eigenvalues of some integral operator in a Hilbert space. This condition can be directly verified, based on the dynamics and cost functional coefficients.

The cheap control problem is an optimal control problem (differential game) with a small control cost (with respect to a state cost) in the cost functional. This problem is of considerable meaning in such topics of control theory as singular optimal control and its regularization (Bell & Jacobson, 1975), limitations of linear optimal regulators and filters (Braslavsky *et al.*, 1999; Kwakernaak & Sivan, 1972), limitations of nonlinear optimal regulators (Seron *et al.*, 1999), high gain control (Kokotovic *et al.*, 1986; Young *et al.*, 1977), inverse control problems (Moylan & Anderson, 1973), guidance problems (Cottrell, 1976; Zarchan, 1994), robust control of systems with disturbances (Turetsky & Glizer, 2004; Turetsky & Glizer, 2007), and some others. Cheap control problems have been investigated extensively for systems with a single decision maker (see e.g. (Kokotovic, 1984) and references therein). More recent results can be found in (Woodyatt *et al.*, 2002; Glizer *et al.*, 2007) and references therein. Cheap controls for differential games have been investigated much less. To the best knowledge of the authors, there are only few works where differential games with cheap control have been studied. In (Starr & Ho, 1969; Glizer, 2000; Turetsky & Glizer, 2004; Turetsky & Glizer, 2007), a finite-horizon game was investigated, while in (Petersen, 1986; Glizer, 2009), an infinite-horizon case was analyzed. In all these works, excepting (Turetsky & Glizer, 2007) the case of the minimizer's cheap control was treated, while in (Turetsky & Glizer, 2007), both the minimizer's and the maximizer's controls were assumed to be cheap.

In the present paper, the general tracking problem is considered. In this problem, a tracking criterion is chosen as a Lebesgue-Stilties integral G of squared discrepancy between the system motion and a given vector function (tracked trajectory), calculated over the mixed discrete-continuous measure. The problem is solved by using an auxiliary LQDG, where the state term of the cost functional is represented by G. Both the minimizer's and the maximizer's controls are cheap. Note that this game is a cheap control version of the LQDG, considered in (Shinar *et al.*, 2008). Novel, cheap control, solvability condition is established. It is shown that, subject to some additional conditions, the optimal cheap control strategy also solves the original tracking problem.

2. TRACKING PROBLEM

2.1. Motivating Guidance Example

Consider a planar engagement between two moving objects (players) - a pursuer and an evader. The schematic view of this engagement is shown in Fig. 1. The X axis of the coordinate system is aligned with the initial line of sight. The origin is collocated with the initial pursuer position. The points (x_p, y_p), (x_e, y_e) are the current coordinates; V_p and V_e are the velocities and a_p, a_e are the lateral accelerations of the pursuer and the evader respectively; φ_p, φ_e are the respective angles between the velocity vectors and the reference line of sight; and $y = y_e - y_p$ is the relative separation normal to the initial line of sight. The line-of-sight angle λ is the angle between the current and initial lines of sight, r is the current range between the objects.

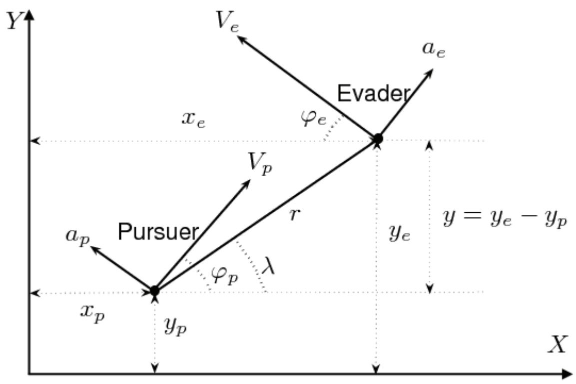

Figure 1: Interception geometry

It is assumed that the dynamics of each object is expressed by a first-order transfer function with the time constants τ_p and τ_e, respectively. The velocities V_p, V_e and the bounds of the lateral acceleration commands $a_p^{\max}$, $a_e^{\max}$ of both objects are constant.

If the aspect angles φ_p and φ_e are small during the engagement then (Shinar, 1981) the trajectories of the pursuer and the evader can be linearized with respect to the nominal collision geometry. The final interception time can be easily calculated as $t_f = r_0/(V_p + V_e)$, where r_0 is the initial distance between the objects and the initial time $t_0 = 0$. The linearized model is described by the following differential equation:

$$\dot{x} = Ax + bu + cv, \quad x(0) = x_0, \tag{1}$$

where the state vector is $x = (x_1, x_2, x_3, x_4)^T = (y, \dot{y}, a_e, a_p)^T$, the superscript T denotes the transposition,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1/\tau_e & 0 \\ 0 & 0 & 0 & -1/\tau_p \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/\tau_p \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 0 \\ 1/\tau_e \\ 0 \end{bmatrix}, \tag{2}$$

$$x_0 = (0, x_{20}, 0, 0)^T, \quad x_{20} = V_e\varphi_e(0) - V_p\varphi_p(0). \tag{3}$$

The controls of the pursuer u and the evader v are the respective lateral acceleration commands.

In this engagement, the pursuer can have different objectives. For example, (1) to intercept the evader, (2) to intercept it with zero relative velocity (rendezvous), (3) to reach some prescribed points during the engagement, (4) to track a prescribed relative separation profile, etc. These objectives are expressed by the following cost functionals:

$$J_1 = x_1^2(t_f), \; J_2 = x_1^2(t_f) + x_2^2(t_f), \tag{4}$$

$$J_3 = \sum_{i=1}^{K} (x_1(t_i) - x_{1i})^2, \; J_4 = \int_0^{t_f} (x_1(t) - y(t))^2 dt, \tag{5}$$

where (t_i, x_{1i}), $i = 1, \ldots, K$, are given points on the plane (t, x_1); $y(t)$ is the prescribed function. Thus, the pursuer's objective becomes to guarantee a small value for the quadratic functionals (4) - (5). In the next section, based on these functionals, a general tracking problem is formulated.

2.2. Robust Tracking Problem Formulation

Consider a controlled system

$$\dot{x} = A(t)x + B(t)u + C(t)v,\ x(t_0) = x_0,\ t_0 \le t \le t_f, \tag{6}$$

where $x \in R^n$ is the state vector, $u \in R^r$ and $v \in R^s$ are the control and the disturbance, respectively, t_0 and t_f are prescribed initial and final time instants; the matrices $A(t)$, $B(t)$ and $C(t)$ are continuous.

Let $t_i \in (t_0, t_f]$, $i = 1,\dots,K$, and $(a_j, b_j) \subset [t_0, t_f]$, $j = 1,\dots,L$, be prescribed time instants and non-intersecting intervals, such that at least one of the conditions $t_K = t_f$, $b_L = t_f$, is satisfied. Let $y(t)$ and $D(t)$, $t \in [t_0, t_f]$, be vector and matrix functions of the dimensions n and $n \times n$, respectively, continuous on each interval $[a_j, b_j]$, $j = 1,\dots,L$. Let define the cost functional

$$J = G(x(\cdot)) = \sum_{i=1}^{K} \left| D(t_i)(x(t_i) - y(t_i)) \right|^2 + \sum_{j=1}^{L} \int_{a_j}^{b_j} \left| D(t)(x(t) - y(t)) \right|^2 dt, \tag{7}$$

where $|\cdot|$ is the Euclidean norm of the vector. Note that the first sum in the right-hand part of (7) is the sum of the intermediary costs, mentioned by Bernhard Bernhard. The functional (7) can be written as a Lebesgue-Stilties integral

$$G(x(\cdot)) = \int_{[t_0, t_f]} \left| D(t)(x(t) - y(t)) \right|^2 dm(t), \tag{8}$$

where the bounded variation function $m(t)$ has the following structure. Let $T = \bigcup_{j=1}^{L} (a_j, b_j)$ and $\zeta(t)$ be an indicator function of T: $\zeta(t) = 1$ for $t \in T$ and $\zeta(t) = 0$ for $t \notin T$. Let $\chi([a,b])$ be the number of the values $t_i \in [a,b]$. Then,

$$m(t) = \int_{t_0}^{t} \zeta(s) ds + \chi([t_0, t]). \tag{9}$$

It is assumed that the disturbance satisfies the integral constraint

$$\int_{t_0}^{t_f} v^2(t) dt \le \nu,\ t \in [t_0, t_f). \tag{10}$$

Problem. For a given $\nu > 0$ and for any given $\zeta > 0$, to construct a feedback strategy $u_\zeta(t, x)$ such that the inequality

$$J < \zeta, \tag{11}$$

is satisfied for any admissible disturbance $v(t)$.

This *robust tracking problem* is treated by embedding it into an auxiliary linear-quadratic differential game.

3. LINEAR-QUADRATIC DIFFERENTIAL GAME

3.1. Game Formulation

For (6), let formulate a differential game with the cost functional

$$J_{\alpha\beta} = J_{\alpha\beta}(u(\cdot), v(\cdot)) = G(x(\cdot)) + \alpha \int_{t_0}^{t_f} |u(t)|^2\, dt - \beta \int_{t_0}^{t_f} |v(t)|^2\, dt, \tag{12}$$

where the functional G is given by (7), the constants α, β are positive.

The objective of the first player ($u(t)$) is to minimize (12), while the second player ($v(t)$) maximizes it, by using feedback strategies $u(t, x)$ and $v(t, x)$, respectively. These strategies are defined for $t \in [t_0, t_f]$, $x \in R^n$. Due to (Krasovskii & Subbotin, 1988) it is assumed that the functions $u(t, x)$ and $v(t, x)$ are measurable w.r.t. t for each fixed x and satisfy the Lipschitz condition w.r.t. x uniformly w.r.t. t. Moreover, for $u = u(t, x)$ and any $v(\cdot) \in L_2^s[t_0, t_f]$, the initial value problem (6) has a solution on the entire interval $[t_0, t_f]$, where $L_2^m[t_0, t_f]$ denotes the space of square-integrable functions $f(\cdot) : [t_0, t_f] \to R^m$. Similarly, for $v = v(t, x)$ and any $u(\cdot) \in L_2^r[t_0, t_f]$, the initial value problem (6) has a solution on the entire interval $[t_0, t_f]$. In the sequel, the sets Υ and ς of all such functions $u(t, x)$ and $v(t, x)$ are called the sets of admissible feedback strategies of the minimizer and the maximizer, respectively. For a given $u(\cdot)\in \Upsilon$, the value

$$J_u(u(\cdot);t_0,x_0)=\sup_{v(t)\in L_2^s[t_0,t_f]} J_{\alpha\beta}, \tag{13}$$

is called the *guaranteed result of* $u(\cdot)$. The strategy $u^0(\cdot)$ is called *optimal* if

$$J_u(u^0(\cdot);t_0,x_0)=\min_{u(\cdot)\in \Upsilon} J_u(u(\cdot);t_0,x_0)\cong J_u^0(t_0,x_0). \tag{14}$$

Similarly the *guaranteed result of the strategy* $v(\cdot)\in \varsigma$ is

$$J_v(v(\cdot);t_0,x_0)=\inf_{u(t)\in L_2^r[t_0,t_f]} J_{\alpha\beta}. \tag{15}$$

The *optimal* strategy $v^0(\cdot)$ is defined by

$$J_v(v^0(\cdot);t_0,x_0)= \max_{v(\cdot)\in\varsigma} J_v(v(\cdot);t_0,x_0)\cong J_v^0(t_0,x_0). \tag{16}$$

If

$$J_u^0(t_0,x_0)=J_v^0(t_0,x_0)\cong J^0(t_0,x_0), \tag{17}$$

then $J^0(t_0,x_0)$ is called the LQDG *value*, and the pair of optimal strategies $(u^0(\cdot),v^0(\cdot))$ is called the LQDG *saddle point*. In this case, the LQDG is called *solvable*, and the triplet $\{J^0(\cdot),u^0(\cdot),v^0(\cdot)\}$ constitutes its solution.

In the sequel, this LQDG is analyzed for the cheap control case

$$\alpha\to 0,\quad \beta\to 0,\quad \alpha/\beta=\mu=const. \tag{18}$$

3.2. Cheap Control Solvability Condition

It can be directly shown that the solvability condition, established by Shinar et al (2008), is not suitable for a cheap control case (18). In this section, a novel condition is formulated.

For a given $t\in[t_0,t_f]$, the function $m(\cdot)$ generates (see (Balakrishnan, 1976)) the Hilbert space $L_2^n([t,t_f],m)$ of the vector functions $f(\cdot):[t,t_f]\to R^n$, square-integrable over $m(\cdot)$ on the interval $[t,t_f]$. The inner product in this space is defined as

$$\langle f_1(\cdot),f_2(\cdot)\rangle_t=\int_{[t,t_f]} f_1^T(\eta)f_2(\eta)dm(\eta), \tag{19}$$

yielding the norm

$$\| f(\cdot)\|_t=\langle f(\cdot),f(\cdot)\rangle_t^{1/2}=\left(\int_{[t,t_f]} |f(\eta)|^2\,dm(\eta)\right)^{1/2}. \tag{20}$$

Let introduce the parametric family of the operators $\Phi_{\alpha\mu}(t)$, $t\in[t_0,t_f]$, mapping $L_2^n([t,t_f],m)$ into itself:

$$\Phi_{\alpha\mu}(t)=\frac{1}{\alpha}\Phi_\mu(t),\ \ \Phi_\mu(t)\cong\mu\Phi_v(t)-\Phi_u(t), \tag{21}$$

where

$$\Phi_k(t)f(\cdot)= \int_{[t,t_f]} F_k(t,\eta,\nu)f(\nu)dm(\nu),\ \eta\in[t,t_f],\ \ k=u,v, \tag{22}$$

$$F_k(t,\eta,\nu)= D(\eta)X(\eta,t_f)\left(\int_t^{\min(\eta,\nu)} Q_k(\tau)d\tau\right)X^T(\nu,t_f)D^T(\nu),\quad k=u,v, \tag{23}$$

$$\begin{aligned} Q_u(t)&=X(t_f,t)B(t)B^T(t)X^T(t_f,t),\\ Q_v(t)&=X(t_f,t)C(t)C^T(t)X^T(t_f,t), \end{aligned} \tag{24}$$

and $X(t,\tau)$ is the fundamental matrix of the homogenous equation $\dot{x}=A(t)x$. Also denote

$$Q_{\alpha\mu}(\tau)=\frac{1}{\alpha}(\mu Q_v(\tau)-Q_u(\tau)). \tag{25}$$

Due to (Shinar *et al.*, 2008), for any $t\in[t_0,t_f]$, the operators $\Phi_u(t)$ and $\Phi_v(t)$ are self-adjoint, positive and compact.

Let introduce the matrix, vector and scalar functions $R_{\alpha\mu}(t)$, $r_{\alpha\mu}(t)$ and $\rho_{\alpha\mu}(t)$, satisfying the following impulsive differential equations

$$\frac{dR}{dt}=-RQ_{\alpha\mu}(t)R-\zeta(t)S(t). \tag{26}$$

$$R(t_f+0)=0,\ \ R(t_i+0)-R(t_i)=-S(t_i), \tag{27}$$

$$\frac{dr}{dt} = \tag{28}$$
$$-R(t)Q_{\alpha\mu}(t)r + 2\zeta(t)X^T(t,t_f)D^T(t)y(t),$$
$$r(t_f+0)=0,$$
$$r(t_i+0)-r(t_i) = \tag{29}$$
$$2X^T(t_i,t_f)D^T(t_i)D(t_i)y(t_i),$$

$$\frac{d\rho}{dt} = -\frac{1}{4}r^T(t)Q_{\alpha\mu}(t)r(t) - \tag{30}$$
$$\zeta(t)y^T(t)D^T(t)D(t)y(t),$$
$$\rho(t_f+0)=0,$$
$$\rho(t_i+0)-\rho(t_i) = \tag{31}$$
$$-y^T(t_i)D^T(t_i)D(t_i)y(t_i),$$

where $i=1,\dots,K$; $\zeta(t)$ is the indicator function of the set T ; $S(t)=X^T(t,t_f)D^T(t)D(t)X(t,t_f)$, $Q_{\alpha\mu}(t)$ is given by (25). Note that this impulsive system consists of the Riccati matrix differential equation (26), the linear vector differential equation (28) and the trivial scalar differential equation (30).

Let $\lambda_{\mu i}(t)$, $i=1,2,\dots$, be the eigenvalues of the operator $\Phi_\mu(t)$.

Theorem 1. *Let for fixed* $\mu>0$,

$$\sup_{t\in[t_0,t_f)}\ \sup_i \lambda_{\mu i}(t)\le 0. \tag{32}$$

Then, for an arbitrary small $\alpha>0$ *and* $\beta=\alpha/\mu$, *the LQDG (6) -- (12) is solvable. For any position* $(t,x)\in[t_0,t_f]\times R^n$, *the value and the saddle point of the LQDG are given by*

$$J^0(t,x)=J^0_{\alpha\mu}(t,x)=$$
$$x^TX^T(t_f,t)R_{\alpha\mu}(t)X(t_f,t)x+ \tag{33}$$
$$r^T_{\alpha\mu}(t)X(t_f,t)x+\rho_{\alpha\mu}(t),$$

$$u^0(t,x)=u^0_{\alpha\mu}(t,x)=-\frac{1}{2\alpha}B^T(t)l^0_{\alpha\mu}(t,x), \tag{34}$$

$$v^0(t,x)=v^0_{\alpha\mu}(t,x)=\frac{1}{2\beta}C^T(t)l^0_{\alpha\mu}(t,x), \tag{35}$$

where

$$l^0_{\alpha\mu}(t,x)\ \square\ X^T(t_f,t)(2R_{\alpha\mu}(t)X(t_f,t)x \tag{36}$$
$$+r_{\alpha\mu}(t)),$$

$R_{\alpha\mu}(t)$, $r_{\alpha\mu}(t)$ *and* $\rho_{\alpha\mu}(t)$ *satisfy (26) - (31).*

4. TRACKING PROBLEM SOLUTION

4.1. Tracking by LQDG Cheap Control

In this section, it is shown how the optimal minimizer strategy $u^0_{\alpha\mu}(\cdot)$ can be used for tracking a given trajectory $y(t)$. In this case, the opponent control $v(t)$ is considered as an unknown disturbance from $L^s_2[t_0,t_f]$. The tracking accuracy is evaluated by the functional (7).

Let $x_{\alpha\mu}(t)$ denote the solution of (6) for $u=u^0_{\alpha\mu}(t,x)$ and $v=v(t)$, and $u_{\alpha\mu}(t)\cong u^0_{\alpha\mu}(t,x_{\alpha\mu}(t))$. Let introduce the operator $\Phi_{\mu 0}=-\Phi_\mu(t_0)$, and the function

$$w_0(t)=D(t)(X(t,t_0)x_0-y(t)),\ t\in[t_0,t_f]. \tag{37}$$

Let the eigenvalues and the eigenfunctions of $\Phi_{\mu 0}$ be λ_k and $f_k(t)$, $k=1,2,\dots$, $t\in[t_0,t_f]$, and the function $w_0(t)$ be represented by the series

$$w_0(t)=\sum_{k=1}^{\infty}w_kf_k(t). \tag{38}$$

Theorem 2. *Let the inequality (32) hold. If* $\lambda_k>0$ *for all* $k=1,2,\dots$, *and*

$$\sum_{k=1}^{\infty}\frac{w_k^2}{\lambda_i}<\infty, \tag{39}$$

then for any $v(\cdot)\in L^s_2[t_0,t_f]$,

$$\lim_{\alpha\to 0,\beta=\alpha/\mu}G(x_{\alpha\mu}(\cdot))=0. \tag{40}$$

Corollary 1. *Let the conditions of Theorem 2 hold and* $v(\cdot)\in L^s_2[t_0,t_f]$ *satisfies the constraint (10). Then for any* $\zeta>0$ *there exists* $\alpha=\alpha(\zeta,v)$ *such that*

$$G(x_{\alpha\mu}(\cdot))<\zeta. \tag{41}$$

This corollary means that, subject to the conditions of Theorem 2, the LQDG optimal strategy $u^0_{\alpha(\zeta,v)\mu}(t,x)$ for sufficiently small α solves the robust tracking problem.

4.2. Control Boundedness

In the previous section, it has been shown that, subject to conditions of Theorem 2, the optimal minimizer strategy $u^0_{\alpha\mu}(t,x)$ solves the tracking problem in the sense (40). However, the corresponding time realization

$u_{\alpha\mu}(t)$ can become unbounded. In some real-life problems, such an unboundedness is inconvenient and even unacceptable. Therefore, deriving the conditions, guaranteeing the boundedness of $u_{\alpha\mu}(t)$, is of a considerable importance.

It can be shown, that if the conditions of Theorem 2 are valid, the control time realization is bounded in the sense of $L_2^r[t_0,t_f]$. Next theorem establishes necessary conditions of boundedness in the sense of $C[t_0,t_f]$.

Let $u_\alpha(t,x)$ be a family of admissible minimizer's feedback strategies, where α is a positive parameter, $t\in[t_0,t_f]$, $x\in R^n$. Let $x_\alpha(t)$ denote the solution of (6) for $u=u_\alpha(t,x)$ and arbitrary but fixed $v(\cdot)\in L_2^s[t_0,t_f]$.

Theorem 3. *Let for any* $v(\cdot)\in L_2^s[t_0,t_f]$,

$$\lim_{\alpha\to 0} G(x_\alpha(\cdot))=0. \tag{42}$$

If, subject to (42), the control time realization $u_\alpha(t)=u_\alpha(t,x_\alpha(t))$ *is bounded, i.e. there exists* $C>0$ *such that* $|u_\alpha(t)|\le C$*, for all* $t\in[t_0,t_f]$ *and sufficiently small* $\alpha>0$ *then: either*

(A)

$$a_1>t_0, \tag{43}$$

or

(B) $a_1=t_0$ *and*

$$x_0-y(t_0)\in \mathrm{Ker}D(t_0). \tag{44}$$

Theorem 3 implies that if the condition

$$a_1=t_0 \text{ and } x_0-y(t_0)\notin \mathrm{Ker}D(t_0), \tag{45}$$

is valid and the optimal minimizer strategy $u^0_{\alpha\mu}(\cdot)$ solves the tracking problem, then its time realization $u_{\alpha\mu}(t)$ is necessarily unbounded.

5. EXAMPLES

5.1. Scalar Illustrative Example

Consider the LQDG for the scalar system

$$\dot{x}=u+v, \tag{46}$$

with the pure integral cost functional

$$J_{\alpha\beta}=\int_0^{t_f}(x(t)-y(t))^2dt+ \alpha\int_0^{t_f}u^2(t)dt-\beta\int_0^{t_f}v^2(t)dt, \tag{47}$$

for which $t_0=0$, $A(t)\equiv 0$, $B(t)=C(t)=D(t)=X(t,t_0)\equiv 1$, $K=0$, $L=1$, $a_1=t_0=0$, $b_1=t_f$, yielding $Q_u(t)=Q_v(t)\equiv 1$. By simple algebra, due to (21) -- (22), this implies that

$$\Phi_{\mu 0}f(\cdot)=(1-\mu)\int_0^{\eta}\int_{\nu}^{t_f}f(\xi)d\xi d\nu. \tag{48}$$

Assume $\mu<1$, which guarantees that this operator is positive. Thus, the Sturm-Liouville problem for the operator (48) can be transformed to the boundary value problem

$$\ddot{g}+\gamma^2 g=0,\ g(0)=0,\ g(t_f)=0, \tag{49}$$

where $g(\eta)\cong\int_0^{\eta}\int_{\nu}^{t_f}f(\xi)d\xi d\nu$, $\gamma^2\cong(1-\mu)/\lambda$, λ is the eigenvalue of (48). From (49), the eigenvalues and the eigenfunctions of (48) are

$$\lambda_k=\frac{(1-\mu)t_f^2}{(\pi/2+\pi k)^2}>0,$$
$$f_k(t)=\frac{1-\mu}{\lambda_k}\sin\sqrt{\frac{1-\mu}{\lambda_k}}t,\ t\in[0,t_f], \tag{50}$$
$$k=1,2,\ldots.$$

The eigenfunctions $f_k(t)$ satisfy

$$\langle f_k(\cdot),f_m(\cdot)\rangle_{t_0}=0,\ k\ne m;$$
$$\|f_k(\cdot)\|_{t_0}^2=\frac{(2k+1)^4\pi^4}{32t_f^3}. \tag{51}$$

Let the tracked function be $y(t)=t$, yielding $w_0(t)=x_0-t$. The coefficients w_k of the series (38) are calculated as

$$w_k=\frac{1}{\|f_k(\cdot)\|_{t_0}^2}\int_0^{t_f}(t_0-\xi)f_k(\xi)d\xi= \frac{16(\pi x_0(2k+1)-2(-1)^k t_f)t_f^2}{(2k+1)^4\pi^4}, \tag{52}$$

implying by (50) that

$$w_k^2 / \lambda_k = \frac{256(2(-1)^k t_f - (2k+1)\pi x_0)^2 (\pi/2 + \pi k)^2 t_f^2}{(1-\mu)(2k+1)^8 \pi^8}, \quad (53)$$

which directly yields (39). Thus, the conditions of Theorem 2 are valid, and the cheap control LQDG optimal strategy solves the tracking problem for

$$G(x(\cdot)) = \int_0^{t_f} (x(t) - t)^2 dt.$$

In Fig. 2, the curve $x = t$ and the trajectories of (46) for decreasing values of α, $\mu = 0.5$, $x_0 = 0$, $u = u_{\alpha\mu}^0(t, x)$ and $v(t) = \sin 10t$, are depicted. It is seen that the smaller is α, the better is tracking.

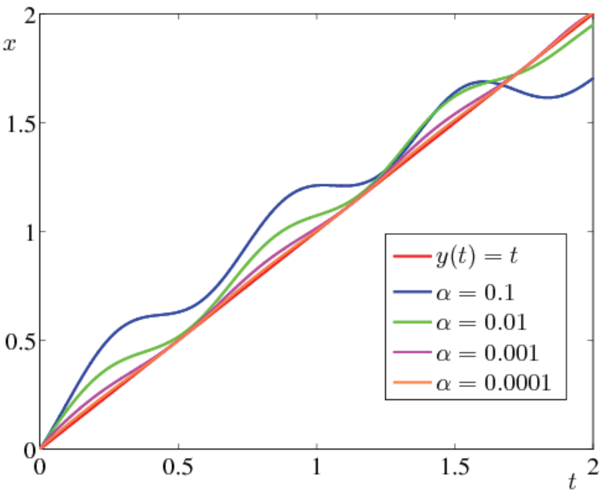

Figure 2: Tracking function $y(t) = t$

The respective control time realizations are presented in Fig. 3. It is seen that in this example the realizations are bounded.

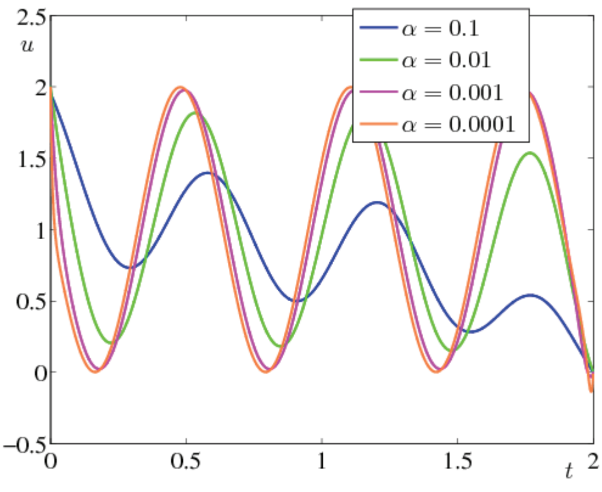

Figure 3: Control time realizations

5.2. Guidance Example

For $t_f = 4$, consider the guidance tracking problem for the system (1) -- (3) with the functional

$$G(x(\cdot)) = [x_1(1) - 1]^2 + [x_1(1.5) - 1.5]^2 + [x_1(1.8) - 3]^2 + x_1(4)^2 + x_2(4)^2 + \int_2^4 [x_1(t) - t(4-t)]^2, \quad (54)$$

which is the particular case of (7). In Fig. 4 and 5, the tracking results (i.e. the graphs of $x_1(t)$ and $x_2(t)$) for $\tau_p = 0.2$ s, $\tau_e = 0.3$ s, $\alpha = 10^{-6}$, $\mu = 0.5$, $v(t) \equiv 100$ m/s^2. It is seen that the system trajectory (shown in blue) tracks accurately both prescribed points and prescribed function (shown in red).

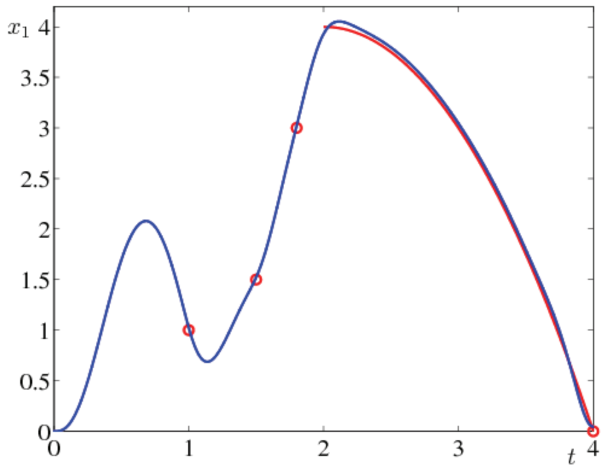

Figure 4: Guidance tracking problem: $x_1(t)$

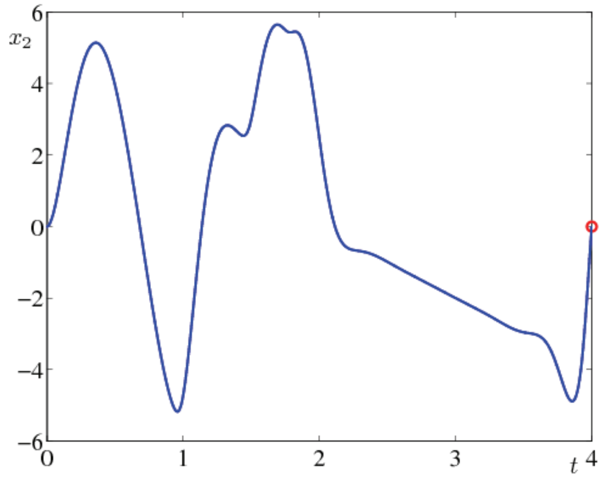

Figure 5: Guidance tracking problem: $x_2(t)$

Remark 1. *In Section 5.1, a relatively simle scalar example was considered. This example admits an analytical solution. For more complex example of Section 5.2, a numerical solution was obtained.*

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