# Algebraic characterization of the invariant zeros structure of LTV bond graph models

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*Abstract*— **In this paper, invariant zeros structure of linear time varying systems modeled by bond graph is derived by using module theory. Infinite structure of the bond graph model is used to get the number of invariant zeros. In the linear time invariant case, null invariant zeros can be directly pointed out. It is no more true for linear time varying models, the combination of graphic and algebraic methods must be considered. A new procedure based on the finite structure of the bond graph model is given to determine the null value of invariant zeros. Algebraic calculations of torsion modules clarify this difference. Based on a simple RLC circuit, different comparative approaches are proposed. A theoretical form based on Jacobson forms of system matrices is proposed and developed with a Maple programm. Some simulations with 20-sim illustrate the results.**

**Keywords: invariant zeros, bond graph, LTV system, module theory**

## 1. INTRODUCTION

Linear systems have been intensively studied since fifty years. Invariant zeros are important for the stability analysis of the controlled systems for several well-known control problems such as the disturbance rejection problem, the input-output decoupling problem and some other problems such as the conception of full order or reduced order observers. Different approaches are proposed, according to the choice of a model, such as state space models, transfer models or graphical representations in case of linear time invariant (LTI) models.

The problems have been tackled under various resolution techniques which are often similar, even if formulations are different. Among these techniques, the structural approach, the algebraic approach and the geometric approach which are popular in control theory, appeared to be very effective. Different steps are often proposed. The first step is mainly at an analysis level (study of the internal structure) and the last step deals with synthesis methods. For linear time varying (LTV) systems, algebraic approach has been proposed and developed by several authors [9], [3]. Nevertheless, the extension to the LTV case is not so easy, even if the problem formulations are similar to the LTI case. From the point view of algebraic approach, a linear system is a finitely generated module over a non commutative polynomial ring of differential operator $\delta$. The issue about system poles/zeros is related to solve differential polynomial equations. In [14] it is shown that a skew polynomial can be written as a product of elementary factors $(\delta-\gamma_i)^{d_i}$. This allows one to give intrinsic definition of the poles and zeros of LTV systems by algebraic approach initiated by Malgrange [13] and Fliess [9].

Because of non commutative properties and derivations of time varying coefficients, the bond graph rules proposed in the LTI case for determining invariant zeros are not valid for the LTV case. Some complementary rules must be added. The invariant zeros structure can be studied with an algebraic approach, but the algebraic calculations are often complex. By combining the two methods, a simple procedure to determine the invariant zeros structure is pointed out. In this paper, null invariant zeros are considered. From the point of view of module theory, a null zero corresponds to the factorization of the term $\delta^n (n \geq 0)$ related to *right roots* of torsion module polynomial representation.

This paper is first concerned with some tools for analysis of LTI bond graph models, first with controllability/observability properties and then with the infinite structure related to input-output causal paths. Then, the procedure to get invariant zeros is explained, from bicausal bond graph models. In section 3, algebraic approach with notions of module and polynomial rings are recalled. From this intrinsic point view, the controllability is related to a certain submodule of system. The procedure to determine invariant zeros structure by using algebraic and bond graph methods is then introduced. A LTV system with several cases is studied. The Jacobson form of system matrices and simulation curves are shown.

## 2. ANALYSIS OF LTI BOND GRAPH MODELS

In a bond graph model, causality and causal paths are useful for the study of properties, such as controllability, observability and systems poles/zeros. Bond graph models with integral causality assignment (BGI) can be used to determine reachability conditions and the number of invariant zeros by studying the infinite structure. In the LTI case, the rank of controllability matrix is derived from bond graph models with derivative causality (BGD). Systems invariant zeros are poles of inverse systems. Inverse systems can be constructed by bond graph models with bicausality (BGB) which are thus useful for the determination of invariant zeros. The number of null invariant zeros can be calculated by studying the infinite structure of BGD models. All these concepts are recalled in this part for LTI bond graph models.

### 2.1. Controllability/Observability

Usually, when studying the solvability conditions and stability conditions for various problems, the controllability and observability properties of the model must be first studied.

The controllability/observability properties have been first derived from a graphical approach using the causality concept in [16]. Controllability conditions on the bond graph representation are recalled.

*Property 1:* [16] A LTI bond graph model is controllable if and only if the two following conditions are verified:

- there is a causal path between each dynamical element and one of the input sources
- Each dynamical element can have a derivative causality assignment in the bond graph model with a preferential derivative causality assignment (with a possible duality of input sources)

### 2.2. Infinite structure

The infinite structure of multivariable linear models is characterized by different integer sets: two sets are recalled here. $\{n'_i\}$ is the set of infinite zero orders of the global model $\Sigma(C,A,B)$ and $\{n_i\}$ is the set of row infinite zero orders of the row sub-systems $\Sigma(c_i,A,B)$. The infinite structure is well defined in case of LTI models [6] with a transfer matrix representation or with a graphical representation (structured approach), [7], and can be easily extended to LTV models with the graphical approach.

The row infinite zero order for the row sub-system $\Sigma(c_i,A,B)$ is the integer $n_i$, which verifies condition $n_i =$ min $\left\{k|c_iA^{(k-1)}B \neq 0\right\}$ . $n_i$ is equal to the number of derivations of the output variable $y_i(t)$ necessary for at least one of the input variables to appear explicitly. The global infinite zero orders [8] are equal to the minimal number of derivations of each output variable necessary so that the input variables appear explicitly and independently in the equations.

The graphical procedure for the determination of the row and global infinite structures of a bond graph model is recalled.

*Definition 1:* The causal path length between an input source and an output detector in the bond graph model is equal to the number of dynamical elements met in the path.

Two paths are different if they have no dynamical element in common.

The order of the infinite zero for the row sub-system $\Sigma(c_i,A,B)$ is equal to the length of the shortest causal path between the $i^{th}$ output detector $y_i$ and the set of input sources. The global infinite structure is defined with the concepts of different causal paths. The orders of the infinite zeros of a global invertible linear bond graph model are calculated according to equation (1), where $L_k$ is the smallest sum of the lengths of the $k$ different input-output causal paths.

$$\begin{cases} n'_1 = L_1 \\ n'_k = L_k - L_{k-1} \end{cases} \tag{1}$$

The study of the infinite structure of LTV bond graph models is quite similar to the LTI case.

### 2.3. Invariant zeros with graphic approach

The number of invariant zeros is determined by the infinite structure of BGI model.

**Assumption 1** In section 2.3, it is supposed that LTI bond graph models are controllable, observable, invertible and square. The state matrix is invertible and the order of the model is $n$.

*Proposition 1:* The number of invariant zeros associated to the bond graph model is equal to $n - \sum n'_i$, where $n$ is the number of state variables and $\{n'_i\}$ is the set of infinite zero orders of the global model.

For bond graph models, invariant zeros equal to zero can be directly deduced from the infinite structure of the BGD model. Some definitions are recalled, [2]. The row infinite structure and the global infinite structure in the BGD are defined with the two sets $\{n_{id}\}$ and $\{n'_{id}\}$

*Property 2:* The infinite zero order $n_{id}$ associated to the $i^{th}$ output variable $y_i(t)$ is equal to the shortest causal path length between the output detector associated to the output variable $y_i(t)$ and the set of input sources in the BGD models.

*Property 3:* The set of infinite zero orders $\{n'_{id}\}$ is obtained with:

$$\begin{cases} n'_{1d} = L_{1d} \\ n'_{id} = L_{id} - L_{(i-1)d} \end{cases} \tag{2}$$

where $L_{id}$ is equal to the smallest sum of $i$ causal path lengths between $i$ output detectors and $i$ input sources (these paths must be different) in the BGD.

*Property 4:* The infinite zero order $n_{id}$ of a LTI bond graph model is equal to the number of null invariant zeros associated to the output variable $y_i(t)$ and the number of null invariant zeros associated to the bond graph model is equal to $\Sigma n'_{id}$.

The previous properties are not always valid in the LTV case. The new procedure related to null invariant zeros is proposed in the sequel.

## 3. ALGEBRAIC APPROACH

For non linear models, variational models can be written with Kahler derivation. These new models are linear time varying (LTV) models. Non linear bond graph models are transformed in LTV bond graph models with some graphical procedures proposed in [1]. In case of the stability analysis, the finite structure must be studied, and in that case LTI extension to the LTV case is not so easy. One approach proposed by [9] for the study of linear systems and then extended to non linear systems for flatness analysis in [10] is the algebraic approach. Different classical problems such as the controllability/observability analysis [5], [12] or input-output decoupling problem [11] are proposed as a direct extension of the LTI case due to some properties of the bond graph representation. This approach is a good solution for studying classical control problems with the stability property on bond graph models. A simple example is first proposed and then the algebraic approach is recalled.

### 3.1. LTV example 1

Consider the bond graph model with integral causality of a LTI system, figure (1). According to Property 1, the first condition is satisfied.

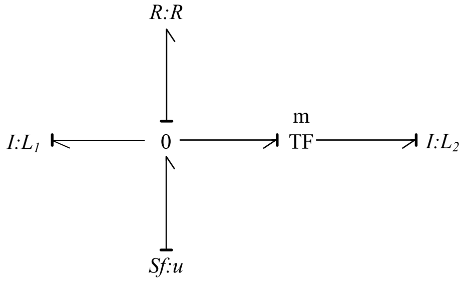


Fig. 1. Bond graph model with integral causality

As shown in figure (2), the second condition for the bond graph model with derivative causality is not verified. So in the LTI case, this system is not controllable. However, one may get a different result if the system becomes a linear time varying model with for example parameter $m(t)$. In that case, the controllability matrix is of full rank. Property 1 gives in that case only sufficient conditions. From a structural point of view, the model is not structurally controllable if some linear relations can be written between the rows of matrix $[A\ \ B]$. These relations can be directly written from a graphical analysis on the bond graph model BGD.

In figure (1), the two state variables are $x_{L_1}$ and $x_{L_2}$. In figure (2), one dynamical element has an integral causality assignment, thus a mathematical relation can be written between state variables, $\dot{x}_{L_1} - m(t)\dot{x}_{L_2} = 0$ (same relation obtained between the rows of matrix $[A(t) \vdots B(t)]$ when applying a structural approach). According to the properties of this equation, the controllability property can be pointed out. The algebraic approach must be applied. If parameter $m = m(t)$, equation $\dot{x}_{L_1} - m(t)\dot{x}_{L_2} = 0$ is not associated to a torsion element, thus this model is controllable. If $m(t) = m$, $\dot{x}_{L_1} - m\dot{x}_{L_2} = 0$ is equivalent to $\delta(x_{L_1} - mx_{L_2}) = 0$, which is the equation of a torsion submodule and in that case the model is not controllable.

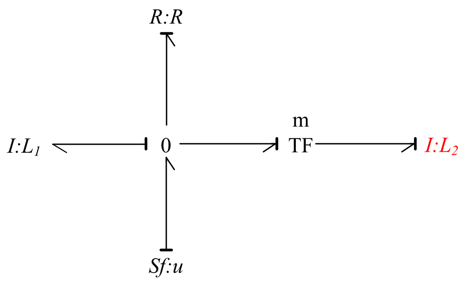


Fig. 2. Bond graph model with derivative causality

The graphical procedure for the study of the observability property is very closed to the one defined in case of the controllability property for LTI models [16]. From a structural point of view, the model is not structurally observable if some linear relations can be written between the columns of matrix $[C^t(t)\ \ A^t(t)]^t$. These relations cannot be written directly from a graphical analysis on the bond graph model. In that case, the concept of duality [15] can be applied.

### 3.2. Module and linear systems

The classical state representation, for a linear time varying system is given by the Kalman form (3), with $x \in \Re^n$, $u \in \Re^m$ and $y \in \Re^p$. Models described in equation (3) are denoted as $\Sigma(C(t), A(t), B(t))$.

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) \end{cases} \tag{3}$$

In Fliess's theoretic approach the systems are the modules. The definitions in this section are the same with the one introduced in [9]. A linear system $\sum$ is a finitely generated left $\mathbf{R}$-module.

*Definition 2:* A(linear) dynamics $\mathscr{D}$ is a system in which a finite set $u = \{u_1, u_2, ..., u_m\}$ of input variables is such that the quotient module $\mathscr{D}/[u]_{\mathbf{R}}$ is torsion. It means that any element in $\mathscr{D}$ can be calculated from $u$ by a linear differential equation.

Equation (3) is equivalent to (4) in a module framework representation, if the output variable is not considered.

$$\begin{pmatrix} I\delta - A(t) & -B(t) \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = R(\delta, t) \begin{pmatrix} x \\ u \end{pmatrix} = 0 \tag{4}$$

The entries of the matrix $R(\delta, t)$ belong to the non commutative ring $\mathbf{R}$ and $\begin{bmatrix} x & u \end{bmatrix}_{\mathbf{R}} \in \Omega$ (here $x$ and $u$ are considered as row vectors, $\Omega$ is the $\mathbf{R}$-module and $\delta = d/dt$).

The structural properties of (4) are then translated to a module framework. The controllability property is thus directly deduced from the Jacobson form of matrix $R(\delta, t)$, or from the torsion module associated to the state equation, if this torsion element exits.

### 3.3. Controllability of LTV systems

From an algebraic point of view, controllability is related to the torsion submodule $\mathscr{T}(\Sigma)$ which is the module of input decoupling zeros, where $\mathscr{T}(\Sigma)$ is the torsion submodule of system module $\Sigma$.

For the controllability analysis [12], in the LTV case, the second condition in property (1) is not a necessary condition. Some differential equations can be written between state variables: if these equations define a torsion element then the model is not controllable, otherwise it is controllable.

*Property 5:* [9] A linear system is controllable iff it is a free $\mathbf{R}$-module, i.e. the torsion submodule is trivial $\mathscr{T}(\Sigma)$=0.

*Definition 3:* [9] The controllability matrix of LTV systems can be written as the form: $\mathscr{C}(\delta, t) = \left[B(t), (A(t) - \delta I_n) B(t), \ldots, (A(t) - \delta I_n)^{n-1} B(t)\right]$. LTV systems are controllable iff $rk(\mathscr{C}(\delta, t)) = n$.

*Example 1:* (continued) For the system with a time varying transformer $m(t)$ shown in figure (1), the state matrix

$A(t)$ and input matrix $B(t)$ are equal to:

$$A(t)=\begin{bmatrix} -\frac{R}{I_1} & -\frac{R}{m(t)I_2} \\ -\frac{m(t)R}{I_1} & -\frac{R}{I_2} \end{bmatrix} \quad B(t)=\begin{bmatrix} 1 \\ m(t) \end{bmatrix}$$

The controllability matrix is

$$\mathscr{C}(t,\delta)=[B(t),(A(t)-\delta I)B(t)]=\left[B(t),A(t)B(t)-\frac{dB(t)}{dt}\right]$$

$$=\begin{bmatrix} 1 & -\frac{R}{I_1}-\frac{R}{I_2} \\ m(t) & -\frac{m(t)R}{I_1}-\frac{m(t)R}{I_2}-\frac{dm(t)}{dt} \end{bmatrix}$$

Compared with the LTI case, the controllability matrix has an additional term $B'(t)$. In the LTI case, the rank of matrix $\mathscr{C}$ is smaller than system order. The LTI system is noncontrollable. With the term $\frac{dB(t)}{dt}$, $rk(\mathscr{C}(t,\delta))=2$ means that the LTV system is controllable.

### 3.4. Invariant zeros with algebraic approach

Considering a LTV system $\Sigma(C(t),A(t),B(t))$ which is a finitely generated module $M$ over the ring $\mathbf{R}=\mathbf{K}[\delta]$. The module $\mathscr{T}(M/[y]_{\mathbf{R}})$ is torsion and is called the module of invariant zeros of $\Sigma$. Let $z$ be a generator of $\mathbf{R}=\mathbf{K}[\delta]$. There exists $Z(\delta)\in\mathbf{R}$ such that $Z(\delta)z=0$ and let $\bar{K}$ an extension of $K$ over which a set of zeros of $Z(\delta)$ can be derived.

*Definition 4:* [4] $M_{iz}=\mathscr{T}(M/[y]_{\mathbf{R}})$ is the module of the invariant zeros of the LTV system. The invariant zeros of the LTV system are the conjugacy classes of the elements of a full set of Smith zeros of $M_{iz}$.

For the LTV system represented by equation (3), the module of the invariant zeros $M_{iz}$ is defined by $P(\delta,t)\begin{bmatrix}\bar{x}\\ \bar{u}\end{bmatrix}$, where $P(\delta,t)=\begin{bmatrix}\delta I-A(t) & -B(t)\\ C(t) & 0\end{bmatrix}$ is the system's matrix and $\bar{x},\bar{u}$ are the images of $x,u$ in module $M_{iz}$. $P(\delta)$ is singular with certain $\delta=\alpha_i$. With these values of $\delta$, for an input $u(t)=u_0e^{\alpha t},t\geq 0$, there exist initial state variables $x_0$ such that the output is null:$y\equiv 0,t\geq 0$.

## 4. MAIN CONTRIBUTIONS

In case of LTV models, the controllability/observability matrices are quite difficult to derive. From a structural point of view, the bond graph approach is simple if the algebraic and structural approaches are combined (see previous sections). For the study of the finite structure and particularly for the invariant zeros which play an essential role in the stability property of the controlled model, solutions are proposed in [17], with the algebraic approach combined with the bicausality assignment on the bond graph model. For classical control problems, invariant zeros must be studied for global models (all input and output variables) and also for row submodels (only one output variable). In [17], it is proved that some uncontrollable parts of the BGB models must be compared from an algebraic point of view (torsion submodules). In this paper the focus is on the invariant zeros with a zero value. For LTI models, the infinite structure of the BGD models is directly related to these particular invariant zeros. For LTV models, conditions are only sufficient and a quite similar extension to the study of the controllability/observability is proposed for the study of null invariant zeros.

### 4.1. Invariant zero: bond graph procedure

**Assumption 2** In section 4, it is supposed that LTV bond graph models are controllable, observable, invertible and square. The state matrices are invertible and the models order is $n$.

*1) Bond graph procedure:* A bond graph interpretation of the procedure for the determination of invariant zeros of LTV systems is implemented in figure (3). Combined with the algebraic method, one can finally get the torsion module concerning the invariant zeros structure. In the first step, BGI model can verify the existence of invariant zeros by proposition (1). Because invariant zeros are the poles of inverse system in the general case, BGB model is utilized to get the inverse model. In the third step, the procedure introduced in [17] is required to get equations of four kinds of elements in BGB model:

- Output detectors (variables are set to a zero value)
- Dynamical elements with a derivative causality
- Input sources
- Dynamical elements with an integral causality

Mathematical relations are written for the unknown variables associated to elements of the BGB model (effort or flow variable depending on the causality assignment). From this set of mathematical relations, the torsion module can be highlighted and some polynomials can be written from dynamical elements with an integral causality. For the last step, some state variables dependent on $u$ are substituted by $u$ to get equations of torsion module $\mathscr{T}(\Sigma/[y]_{\mathbf{R}})$. And the torsion module has the form:

$$P(\delta)\xi=0,\quad P(\delta)=\delta^n+\sum_{i=1}^{n}a_i\delta^{n-i} \tag{5}$$

where $P(\delta)$ is a differential polynomial and equation (5) is the generate equation of torsion module $\mathscr{T}(\Sigma/[y]_{\mathbf{R}})$. $\xi$ is a generator of torsion module, in this case, it is equal to $u$.

A new proposition of null invariant zeros of LTV systems is given.

*Proposition 2:* The null invariant zeros for a row LTV bond graph model with Assumption 2 can be derived from BGD model. Considering a LTV bond graph model BGD with an input-output causal path length greater or equal to 1, there exist null invariant zeros if there is no time varying dynamic element in the causal path. Otherwise, in the case of existence of time varying elements in causal path, there still exist null invariant zeros if the differential polynomial equation of torsion module $\mathscr{T}(\Sigma/[y]_{\mathbf{R}})$ has the form: $G(\delta)u=0\rightarrow G'(\delta)\delta^{\tilde{n}_{id}}u=0$, where $G^{(')}(\delta)$ are polynomials of $\delta$ and $\delta^{\tilde{n}_{id}}$ is the right factor of $G(\delta)$. $G(\delta)$ is the input-output causal path gain and $\tilde{n}_{id}$ is the number of null invariant zeros. $G'(\delta)\in\mathbf{R}$ has the form $\sum_{i=0}^{n}a_i\delta^i,a_i\in K$, where $a_0\neq 0$.

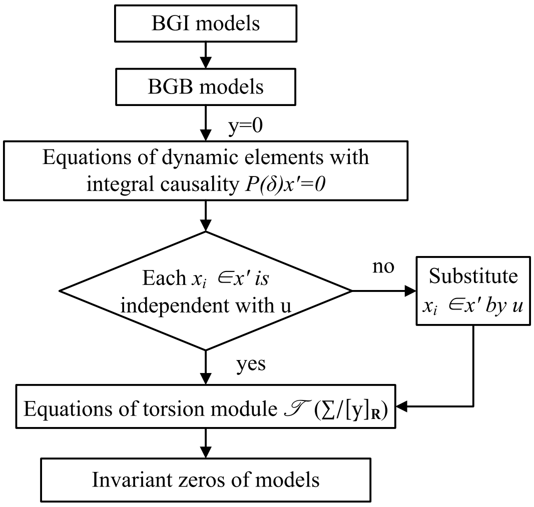


Fig. 3. Procedure for invariant zeros of linear systems

### 4.2. Examples

In this section, the invariant zero structure of a SISO LTV RLC circuit is studied by algebraic and bond graph approaches. The bond graph model of circuit is shown by figure (4). Table (I) gives numerical values of LTI system components.

TABLE I
NUMERICAL VALUES OF RLC CIRCUIT COMPONENTS

| Input $u$ | element $I$ | element $R$ | element $TF$ | element $C$ |
|---|---|---|---|---|
| $1\ V$ | $1\ H$ | $1\ \Omega$ | 2 | $1\ F$ |

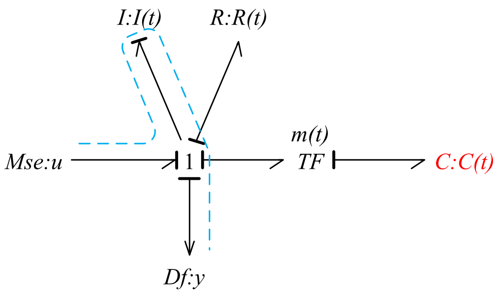


Fig. 4. Bond graph model with integral causality of RLC circuit

The infinite structure of the BGI model, figure (4), is defined as $n = 1$ (causal path $Df \rightarrow I(t) \rightarrow MSe$). By proposition (1), there exist one invariant zero.

Figure (5) gives the bond graph model with derivative causality. The model is controllable. There is an input-output causal path ($Df \rightarrow C(t) \rightarrow MSe$). If the system is time invariant, there is a null invariant zero. In case of a LTV system, the causal path gain must be studied. By proposition (3), the input-output causal path gain is equal to $\frac{1}{m(t)}\delta\frac{C(t)}{m(t)}$. If the coefficient $\frac{C(t)}{m(t)}$ is a constant, i.e. $C(t)$ and $m(t)$ are proportional, there exist a right root of differential polynomial which is a null invariant zero, otherwise, the value of the invariant zero is not equal to zero. This result will be confirmed by calculation and with a simulation in the sequel.

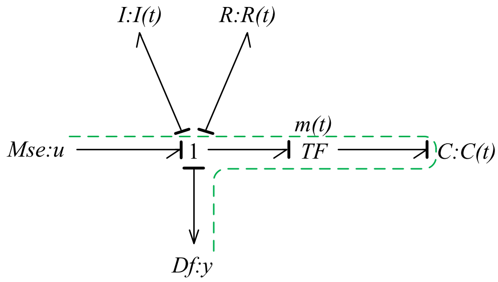


Fig. 5. Bond graph model with derivative causality of RLC circuit

Now the procedure for the determination of the torsion module with the bond graph model with bicausality (BGB) defined in [17] is used. In this simple example, it is not useful, but it will prove the first conclusion obtained from the study of the input output causal path and above all, it is a simple way for the study of torsion submodules associated to any kind of invariant zeros.

In figure (6), the bicausal path is drawn between the input source $Mse : u$ and the output detector $Df : y$. In this simple example, the element $C(t)$ is associated to the torsion module $\mathscr{T}(\Sigma/[y]_{\mathbf{R}})$. The torsion module is the non controllable part of inverse system. The element $C(t)$ (more precisely the state variable) is not controllable, because it is not reachable when the bicausal path is eliminated. Now the procedure to derive the torsion module is given.

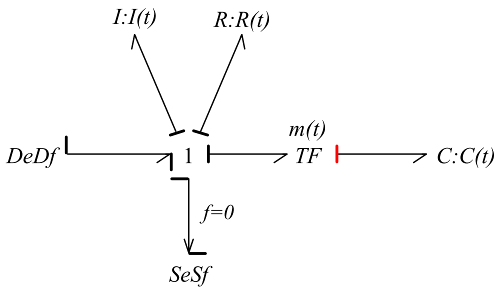


Fig. 6. Bond graph model with bicausality of RLC circuit

**Step 1: output variable**

For the output detector, the flow at the 1 junction is equal to zero. The equation of output variable is $y = \frac{1}{I(t)}x_1$. One relation is thus rewritten: $y = 0$, thus $x_1 = \dot{x}_1 = 0$.

**Step 2: element with derivative causality**

Element $I : I(t) \longrightarrow f_{I_1} = \frac{p_{I_1}}{I_1} = \frac{x_1}{I_1} = 0$

**Step 3: input source**

Source $MSe : u \longrightarrow u = \dot{x}_1 + e_R + \frac{m(t)}{C(t)}x_2 = \frac{m(t)}{C(t)}x_2$

**Step 4: element with an integral causality**

Element $C : C(t) \longrightarrow \dot{x}_2 = f_{C_5} = 0$

**Step 5: expression of torsion module**

In this step, previous equations must be used. According to the length of the input-output causal path, there is only one invariant zero and the degree of the polynomial equation is equal to 1. Equations obtained in step 4 must be redefined according to the previous one. In step 4, one has $\delta x_2 = 0$. Substituting $x_2$ by $u$, it comes

$$\delta \frac{C(t)}{m(t)} u = \left( \frac{C(t)}{m(t)} \delta + \left( \frac{C(t)}{m(t)} \right)' \right) u = 0 \qquad (6)$$

which is the equation of torsion module $\mathscr{T}(\Sigma/[y]_{\mathbf{R}})$. According to Section 3.4, the invariant zero of system is

$$\Delta_{\mathbb{C}(t)} \left( \frac{C(t)m'(t) - C'(t)m(t)}{C(t)m(t)} \right) \qquad (7)$$

which is the conjugacy class of a full set of Smith zeros of $M_{iz}$. So if parameters $C(t)$ and $m(t)$ are constant or proportional, there exist a null invariant zero because of $\delta u = 0$, otherwise the value of the invariant zero is different of zero, which has already be proved.

### *4.3. Several cases*

In this section, several cases related to different time varying parameters of the studied system are considered. With different expressions of input source, one can verify if the invariant zero is null. First, the case without time varying element in causal path of BGD model is studied. Simulation results show that there is no influence for the existence of null invariant zero. Then, with time varying elements in the causal path, three situations are proposed. Firstly, there exists only one time varying element $C(t)$ or $m(t)$, then parameters $C(t)$ and $m(t)$ are not proportional. Finally, the case with one time varying element $I(t)$ outside the causal path and one time varying element $C(t)$ in the path will be studied. In each case, the Jacobson form of the system matrix $P(\delta)$ is given to illustrate the simulation result by algebraic method.

*1. Without time varying elements in the causal path*

The existence of null invariant zeros can be verified if there is no time varying elements in the causal path. Elements $I$ and $R$ with time varying parameters are considered respectively.

*1). Inertial element I is time varying*

Let $I(t) = t^2 + 1$, the invariant zero of system is $\Delta_{\mathbb{C}(t)}(0)$ according to equation (7). So the system has a null invariant zero. The curve of output $Df$ is shown in figure (7) and the output variable is equal to 0 in the steady state part of the curve.

*2). Resistive element R is time varying*

Let $R(t) = t + 1$, the invariant zero of system is $\Delta_{\mathbb{C}(t)}(0)$ according to equation (7). So the system has a null invariant zero. The curve of output $Df$ is shown in figure (8).

*3). Two element I and R are simultaneously time varying*

Let $I(t) = t^2 + 1$ and $R(t) = t + 1$, the invariant zero of system is $\Delta_{\mathbb{C}(t)}(0)$ according to equation (7). So the system has a null invariant zero. The curve of output $Df$ is shown in figure (9).

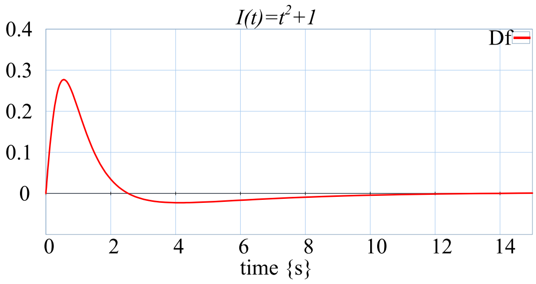


Fig. 7. The output $Df$ curve with $I = t^2 + 1$

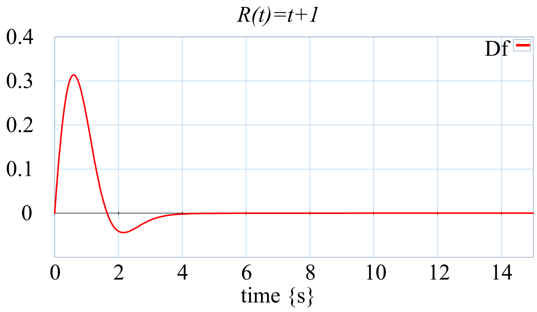


Fig. 8. The output Df curve with $R = t + 1$

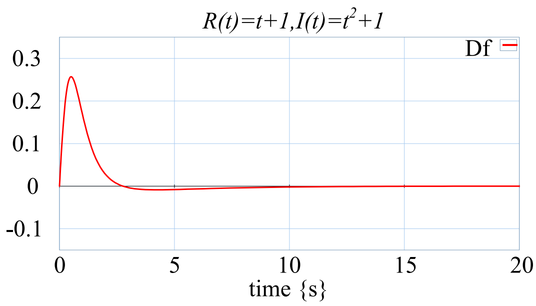


Fig. 9. The output Df curve with $R = t + 1, I = t^2 + 1$

From these three situations, one can verify the proposition (3): if there is no element with time varying parameter in causal path of BGD model, the LTV system has null invariant zero if the causal path length is equal to 1.

*2. Time varying elements in the causal path*

There are two elements $C$ and $TF$ in the causal path of BGD model. Four situations are considered here.

*1). The transformer element TF is time varying*

Let $m(t) = \frac{1}{t+1}$, the invariant zero of system is $\Delta_{\mathbb{C}(t)}\left(-\frac{1}{t+1}\right)$ according to equation (7). So there is no null invariant zero for the system. The curve of output $Df$ is shown in figure (10).

If $m(t) = sint(t) + 2$, the curve of output $Df$ which is not stable is shown in figure (10). As for the instability of the curve, one should calculate if there is(are) unstable pole(s) of system. From algebraic approach, poles are related to the module $\Sigma/[u]_{\mathbf{R}}$. Poles of system can be derived from the Jacobson form of matrix $D(\delta) = (\delta I - A)$. For poles and stability property of LTV systems, [14] is recommended.

*2). The element C is time varying*

Let $C(t) = t + 1$, the invariant zero of system is

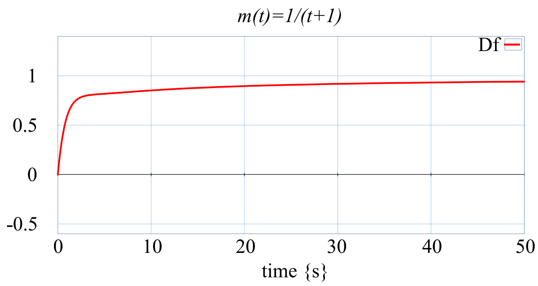


Fig. 10. The output Df curve with $m(t) = \frac{1}{t+1}$

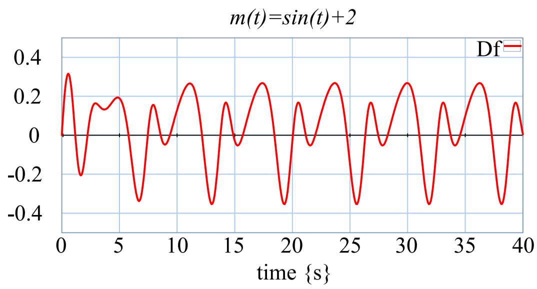


Fig. 11. $m(t) = sin(t) + 2$

$\Delta_{\mathbb{C}(t)}\left(-\frac{1}{t+1}\right)$ according to equation (7). So there is no null invariant zero for the system. The curve of output $Df$ is shown in figure (12).

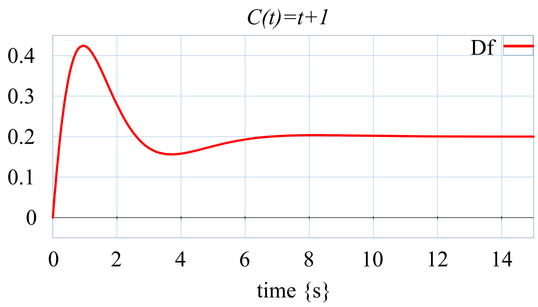


Fig. 12. The output Df curve with $C = t+1$

*3). Two elements TF and C are time varying and proportional*

Let $C(t) = m(t) = t^2 + 1$, the invariant zero of system is $\Delta_{\mathbb{C}(t)}(0)$ according to equation (7). So there is a null invariant zero for the system. The curve of output $Df$ is shown in figure (13).

*4). Two elements TF and C are time varying and non proportional*

Let $m(t) = t+2, C(t) = t^2+1$, the invariant zero of system is $\Delta_{\mathbb{C}(t)}\left(\frac{-t^2-4t+1}{(t^2+1)(t+2)}\right)$ according to equation (7). So there is no null invariant zero for the system. The curve of output $Df$ is shown in figure (14).

From these four aforementioned examples, one can find that there exists invariant zero in spite of the existence of time varying elements in the causal path of BGD model. It means that the rule to determine null invariant zero from the BGD model must be completed with a study of the BGB model and an algebraic criteria. The invariant polynomials of system matrices give the intrinsic implementation of the existence of null invariant zeros.

*3. With time varying elements*

The model with two time varying elements $I$ and $C$ is considered here. One is in the input-output causal path and another is not. Let $I = t+1, C = t^2+1$, the invariant zero of system is $\Delta_{\mathbb{C}(t)}\left(\frac{-2t}{t^2+1}\right)$ according to equation (7). So there is no null invariant zero. Figure (15) shows the output $Df$ curve.

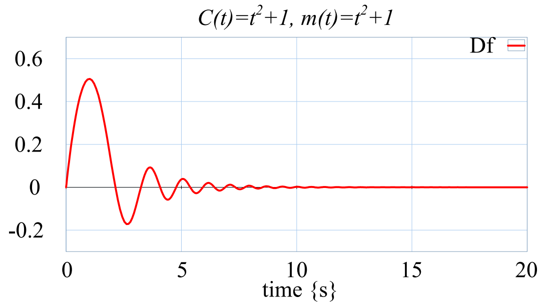


Fig. 13. The output Df curve with $C(t) = m(t) = t^2 + 1$

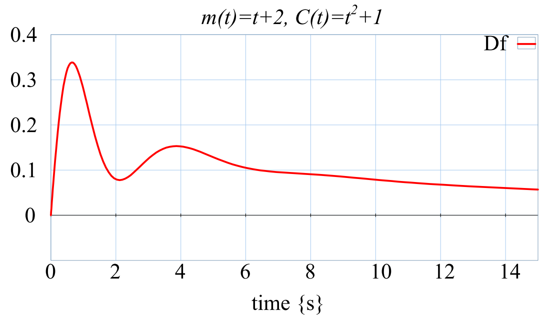


Fig. 14. The output Df curve with $m(t) = t+2, C(t) = t^2+1$

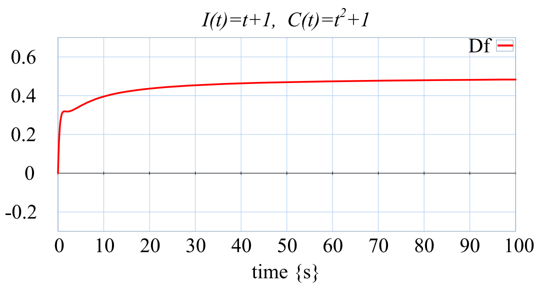


Fig. 15. The output Df curve with $I = t+1, C = t^2+1$

According to the simulation results of aforementioned cases, one can find that there is always one invariant zero. Only the cases where elements $C(t)$ and $m(t)$ are time varying separately or simultaneously but nonproportional have one non null invariant zero. In other cases, there is always a null invariant zero which can not be affected by the time varying parameters of elements. These results verify the algebraic procedure in the article.

## 5. CONCLUSION

In this paper, invariant zeros structure of LTV systems is studied. The LTI bond graph criteria is extended to the time varying case. The rule for determining systems null invariant zeros is developed by using bicausal bond graph models and torsion module notion. The algebraic calculations supported by Maple show the consensus. 20-sim verified the counterparts from simulations.

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