USING LINEAR PROGRAMMING FOR THE OPTIMAL CONTROL OF A CART-PENDULUM SYSTEM

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ABSTRACT
This paper discusses the use of linear programming for the optimal control of a cart pendulum system. The objective function and the constraints are designed to minimize the control effort and the time duration of the operation. Simulations and experimental tests were performed. Restrictions of null angle and angular velocity at the extremes were incorporated in the design specification as well as other physical constraints. In order to compensate for the modeling errors and disturbances, the optimal trajectory was kept within a prescribed precision by means of a closed loop system. The obtained results illustrate that the technique is simple, powerful and always conclusive.

Keywords: Linear Programming, Optimal Control, Anti-oscillatory Control.

1. INTRODUCTION
The problem of optimal control of cranes has been receiving attention from the scientific community because of undeniable practical relevance. As pointed out by Sorensen, Singhose, and Dickerson (2007), the control schemes developed in the literature may be grouped into three categories: time-optimal control, command shaping, and feedback control. Their paper addresses the perturbations by feedback control and, since the trajectories are not known a priory, they used a feed-forward control in the form of an input shaping in order to reduce the motion-induced oscillation.

Several studies (Cheng and Chen, 1996; Auernig and Troger, 1987; Cruz, Leonardi, and Moraes, 2008; Chen, Hein, and Wörn, 2007; Nassif, Domingos, and Gomes, 2010; Garrido *et al*. 2008; Lee, 2004) address the problem of minimum time and differ, for example, with respect to the model utilized, the constraints imposed and the performance index optimized.

In load transfer operations by a crane, a major problem is optimizing the movement from origin to destination, satisfying constraints related to the equipment and to the kinematics of the movement. The carrier may be considered primarily as a cart-pendulum system, where the length of the pendulum is usually variable, representing the lifting.

One difficulty in solving optimal control problems such as the optimal load transfer by a crane is the necessity of solving a two-point boundary value problem, i.e., with constraints on the initial and final states. For instance, a linear quadratic regulator generates an optimal control law but the final state cannot be pre-determined. This limitation is discussed e.g. by Bemporad, Borelli, and Morari (2002) and by Blanchini (1994).

This work addresses the problem for known trajectory boundaries, which is typical for ship unloading operations. For this kind of problem it is highly desirable to have a motion planning scheme that ensures swing reduction and minimum time operation. Feedback control is used to reduce external perturbations and the optimal control trajectory is obtained by solving a simple Linear Programming problem. Thus, the physical constraints can be included explicitly in the design. In a manner similar to the proposed by Sorensen, Singhose, and Dickerson (2007), the cart kinematics is determined by means of an independent feedback control.

This paper discusses the use of Linear Programming as an alternative for solving this type of optimal control problem, assuming that the system dynamics are linear in the state space in the discrete time domain. In this scenario the discrete values of the control vector are the free design variables and the state vector at any sampling time may be written as a linear combination of the control vector and the initial condition. This results in the standard structure of a Linear Programming (LP) problem.

A cart-pendulum lab system was considered to illustrate the proposed approach. The movement cycle begins and ends at given positions and the load is at rest in both, the beginning and end of cycle. Moreover, in the application considered here, the lifting takes place at the beginning and end of the cycle with the cart stationary, i.e., there is no lift during cart movement.

A more efficient strategy is obviously to perform lifting and cart translation simultaneously, but this work intended to show the potential of the methodology, applying it to a lab-scale system that has no motorized lifting.

2. METHOD
The optimal control problem of a dynamical linear system in the discrete time state space can be written in the form of a standard LP problem.

2.1. Linear Dynamics as LP Constraints

Consider the dynamical system in discrete time with a constant sampling time T and described in the state space

$$x_{(k+1)} = A\,x_{(k)} + Bu_{(k)} \tag{1}$$

For any sampling time nT we can write

$$x_{(n)} = A^n\,x_{(0)} + A^{n-1}Bu_{(0)} + A^{n-2}Bu_{(1)} + \cdots$$
$$\cdots + A^1 Bu_{(n-2)} + A^0 Bu_{(n-1)}$$

$$x_{(n)} = F\,x_{(0)} + G\,U \tag{2}$$

where

$$F = A^n$$
$$G = \left[A^{n-1}A^{n-2} \cdots A^1 A^0 \right] diag[BB \cdots BB]$$
$$U = \left[u_{(0)}\,u_{(1)} \cdots u_{(n-2)} u_{(n-1)} \right]^T$$

Note that it is possible to represent the dynamic model as constraints in the form of $AX = B$, which may include the initial conditions $x_{(0)}$ and the final conditions $x_{(n)}$ at the nT instant

$$GU = x_{(n)} - Fx_{(0)}$$
$$\mathbf{A}X = \mathbf{B} \tag{3}$$

where $\mathbf{A} = G$, $X = U$ and $\mathbf{B} = x_{(n)} - F\,x_{(0)}$.

Note that the system dynamics was represented by linear constraints on the control vector. For state constraints in the form of inequalities of type $x_{(m)} \geq \eta$, we can write $x_{(m)} = F_1\,x_{(0)} + G_1\,U_1$. Thus,

$$F_1\,x_{(0)} + G_1\,U_1 \geq \eta$$
$$G_1 U_1 \geq \eta - F_1\,x_{(0)}$$
$$A_1 X_1 \geq B_1 \tag{4}$$

To completely define the LP problem it remains to define a cost function which is linear on states and controls. The choice of cost function depends on the optimization problem to be solved. For example, one can maximize the average speed or minimize the fuel consumption to travel a given distance.

The objective function can be adapted to the particular optimization problem to be solved. A possible objective in optimal control problems is minimizing the sum of the absolute control values at each sampling time. Another possibility is maximizing the average speed to indirectly solve the minimum time problem. These and other objective functions are easily written in the standard form of a Linear Programming problem, i.e., as a linear combination of the control vector.

2.2. Mechanical Model

A scheme of the cart-pendulum system used is shown in Fig. (1), where m_T is the cart mass, m_L the load mass, x_T the cart position and ϕ the load angle.

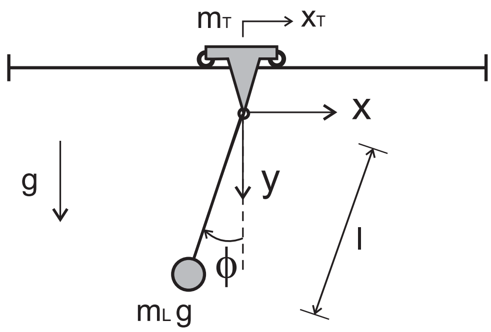

Figure 1: Cart-pendulum scheme

The equations of motion describing the dynamics of the cart-pendulum model were derived using the Newton-Euler formalism as described in Schiehlen (1997) yielding

$$-\ddot{x}_T \cos\phi + \ddot{\phi} l = -g sen\phi - \frac{\bar{c}\dot{\phi}}{m_L}, \tag{5}$$

where g is the gravity acceleration and $\bar{c}$ a damping constant. In handling anti-oscillatory problems, it is expected that the maximum oscillation angle be small (<10º). This condition leads to the approximations $sen\phi \approx \phi$ and $\cos\phi \approx 1$. These approximations simplify the equations of motion to

$$-\ddot{x}_T + \ddot{\phi} l = -g\phi - \frac{\bar{c}\dot{\phi}}{m_L}. \tag{6}$$

2.3. Optimal Control

The objective function chosen here is the control effort in the form of the sum of the absolute control values $|u_1| + |u_2| + \cdots + |u_n|$.

The standard LP formulation admits a single objective function but we propose an approach to minimize both the control effort and the time of operation. The minimum time is obtained by solving a series of minimal-effort LP problems with decreasing final time until constraints can no longer be satisfied. From (6) and defining $x_1 = \phi$, $x_2 = \dot{\phi}$, $x_3 = x_T$, $x_4 = \dot{x}_T$, $c = \frac{\bar{c}}{m_L}$ and $u = \ddot{x}_T$ as the control variable, we get

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ -\frac{g}{l} & -\frac{c}{l} & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{l}\\ 0\\ 1\end{bmatrix}u \tag{7}$$

with the initial condition $x(t_0)=[0 \quad 0 \quad 0 \quad 0]^T$ and final condition $x(t_f)=[0 \quad 0 \quad 0.25 \quad 0]^T$. Based on the limitations of the real plant we have limited $u_{max}=0.9\,m/s^2$.

Defining the minimum time $t = t_f$ and assuming that t_f is given, consider a problem of minimizing the following functional subject to the two boundaries constraints and denote by $S(t_f)$ this minimum-effort optimal control problem.

$$\min_u |u_1|+|u_2|+\cdots|u_n| \tag{8}$$

The following algorithm solves a series of $S(t)$ problems with a tolerance $\epsilon>0$, were x is an arbitrary real value, such that $x>t_f$.

```
t1 = 0
t2 = x
while |t2-t1| > ϵ
   t = (t1+t2)/2
   if S(t) exists
       t2 = t
   else
       t1 = t
   end
end
tf = t2
u* = S(tf)
```

Thus u^* solves $S(t_f)$ with a minimum time t_f. In other words, it is a solution for simultaneous minimum time and minimum effort problem.

2.4. Testing Apparatus

In order to validate the numerical results and implement the proposed control law, it was used a Bytronic lab equipment that allows inverted pendulum or simple pendulum experiments. The schematic diagram of the equipment is shown in Fig. 2.

The pendulum consists of a *0.215 kg* mass connected to the cart by a rod. The mass can be fixed on the rod at different distances from the cart. The cart driver has a position control system with speed compensation. In this loop there is access to the reference signal and the cart position and speed signals.

There is also access to the pendulum angular position signal (not shown in the figure). Since the implementation details of this internal control system are not well documented we chose to consider this as part of the cart sub-system and its transfer function was experimentally identified via step excitation. A second order transfer function was selected as

$$P(s)=\frac{K_n\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{9}$$

and the parameters $K_n=0.025$, $\omega_n=31rd/s$, $\zeta=0.35$ were determined. The gain values for the speed and cart position sensors are $Ktaco=0.25V/m/s$ and $Kpot=0.06V/degrees$, respectively.

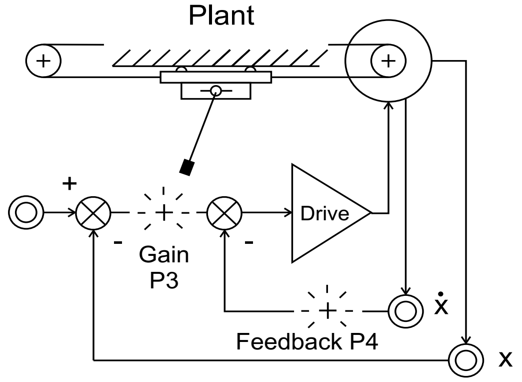

Figure 2: Schematic diagram of the equipment

3. RESULTS

3.1. Simulation

The state equation was discretized with a sampling time $T=15\,ms$ and the optimal control vector was obtained for N sampling periods and including the two boundary constraints. The rod length was set to $l = 0.124m$ and after simple binary search, we found $N=74$ as the minimum number of sampling periods that still leads to a feasible solution. The results are presented in Fig. 3.

3.2. Experimental Results

The optimal control problem was formulated considering the cart acceleration as the manipulated variable. So a way to impose the cart kinematics in the presence of modeling errors and disturbances is necessary. This was achieved through a state feedback control system with a feed-forward action for the acceleration as illustrated in the block diagram of Fig. 4. Note that, although there is a control loop, the optimal control itself is open loop, since there is no feedback for the angle trajectory.

The control system shown in Fig. 4 has three references that are consistent with each other - position, velocity and acceleration of the cart. The position and velocity are states and, therefore, their references apply to the loop, while the desired acceleration enters as a feed-forward action through a block that contains the inverse plant model. The gains $\bar{K}_1$ and $\bar{K}_2$ are the gains from state feedback and were tuned interactively in order to obtain good tracking of the reference signals and for disturbances rejection.

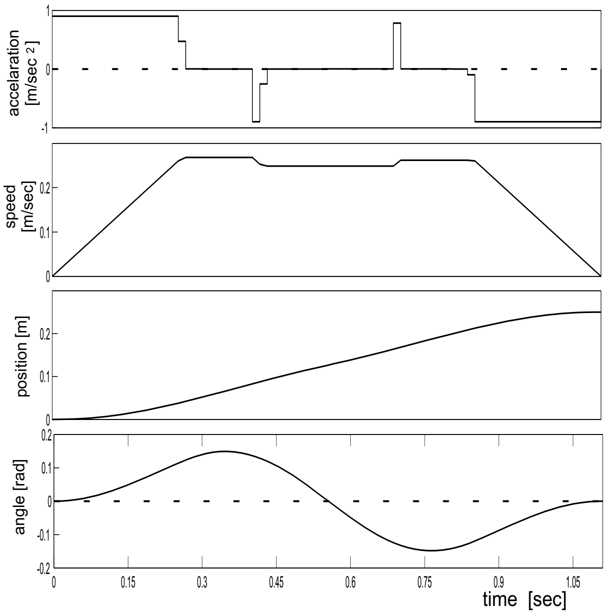

Figure 3: Simulation with *74* sampling periods

Because this control loop cannot be perfect, the optimal control problem will contain errors because the cart acceleration will never be imposed with an infinite precision. The control system along with the trajectory generation and data acquisition, were implemented in Simulink in real time through a data acquisition board and Matlab Windows Target.

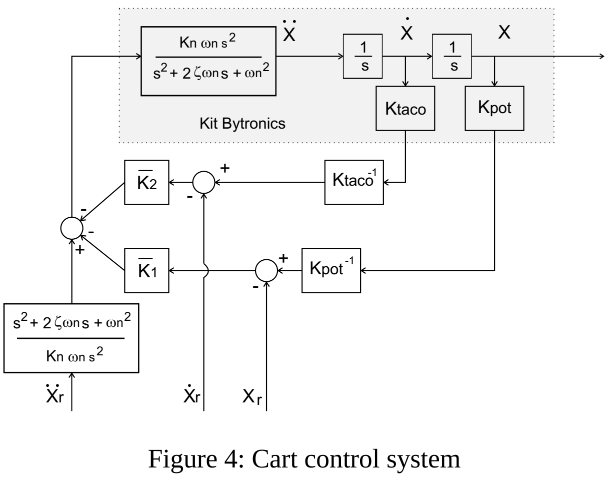

Figure 4: Cart control system

Fig. 5, 6 and 7, show a comparison between the computed and measured position, speed and pendulum angle, respectively. Thin lines were used for generated signals and thick lines for the real signals. Since the experimental apparatus does not have an acceleration sensor for the cart, the real acceleration is not compared with the optimum computed acceleration in none of the following cases.

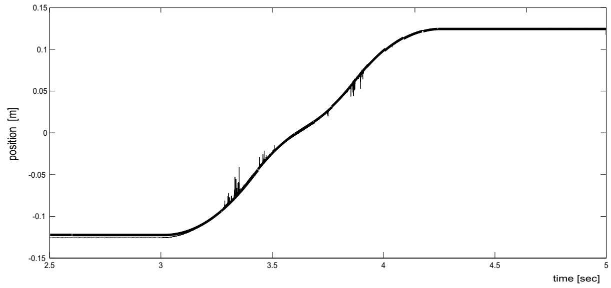

Figure 5: Reference signal and actual cart position

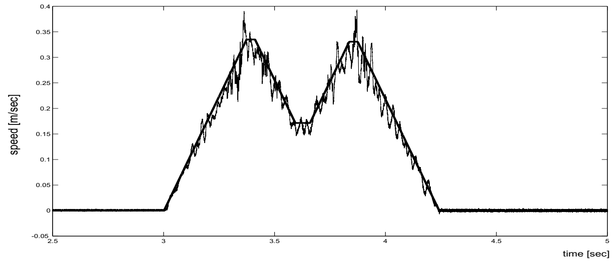

Figure 6: Reference signal and actual cart speed

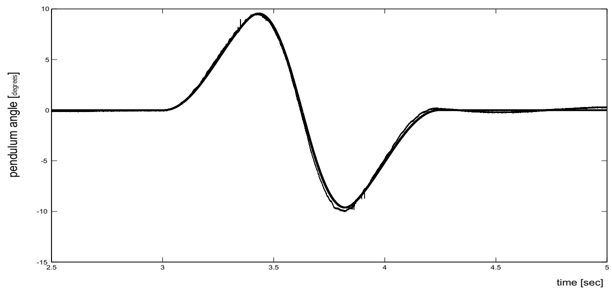

Figure 7: Reference signal and actual pendulum angle

3.3. Sensitivity

A change imposed on the plant model, such as a new mass position, modifies its response. Using the same optimal control vector obtained for *l=0.24m* we evaluate the sensitivity of the system response to variations in rod length *l*. Starting from *l = 0.24 m* changes of length of $\pm 15\,mm$ and $\pm 30\,mm$ in the mass position were performed and the results compared to the optimal trajectory, while keeping the optimal control vector calculated for $l = 0.24\,m$. Figs. 8 to 11 show the effect of changing the rod length on pendulum angle trajectory.

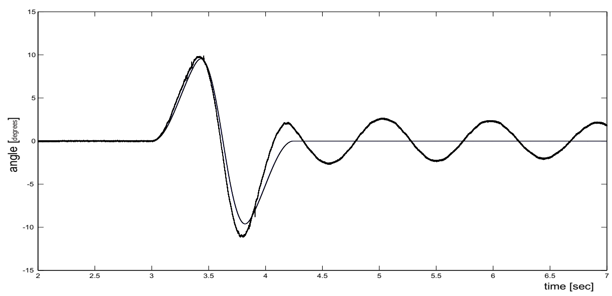

Figure 8: Optimal response for $l = 0.24m$ and measured response $l = 0.21m$

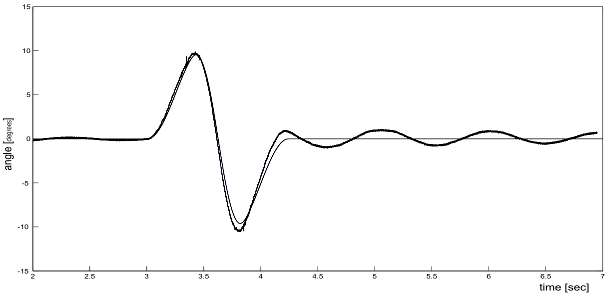

Figure 9: Optimal response for $l = 0.24m$ and measured response $l = 0.225m$

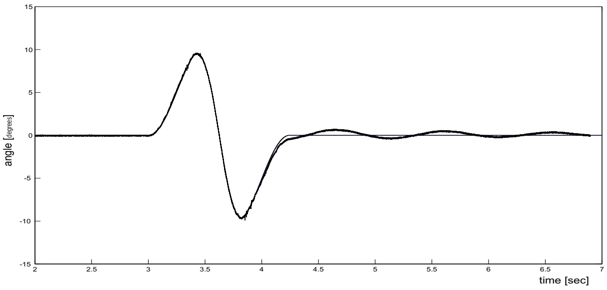

Figure 10: Optimal response for $l = 0.24m$ and measured response $l = 0.255m$

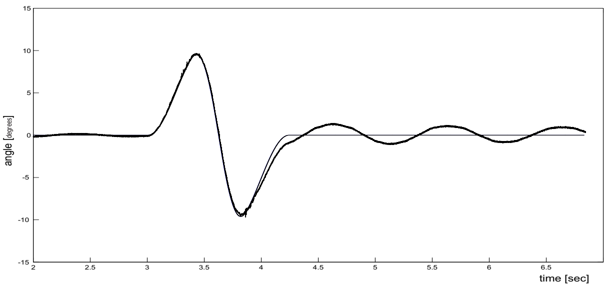

Figure 11: Optimal response for $l = 0.24m$ and measured response $l = 0.27m$

The results suggest that this optimal control problem is sensitive to modeling errors. That is, if the plant model is not well known the optimal trajectory will not be assured. This was expected as the optimal control is running with no feedback.

3.4. Closed Loop Control

To make the system less sensitive to the modeling errors and disturbances, a closed loop optimal control strategy was used. To do so, the angular optimal trajectory is used as a reference for a feedback control system applied to the pendulum angle. Thus, the optimal control signal acts as a feed-forward action and this control loop does just the corrections of deviations from the optimal trajectory. Note that modeling errors such as those arising from considering $sin\phi \approx \phi$ and $\cos\phi \approx 1$ are also reduced by the feedback control.

Since the positioning of the cart is made by means of three references, it is preferable to work directly with the manipulated variable. The diagram of Fig. 12 shows the complete control system, i.e., the state feedback loop for the cart positioning and the closed loop control for the pendulum angle. The transfer function used as the angle controller was $G_C(s) = 25/s+1$ and its parameters were tuned interactively to produce the best insensitivity to the variations of length *l*.

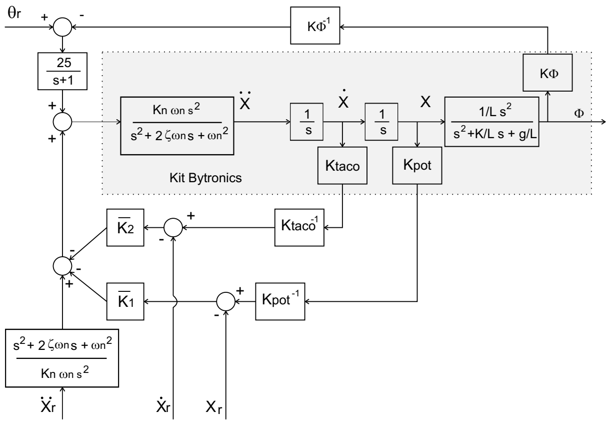

Figure 12: Complete control system

For evaluation of differences between the two methods, data was collected in both closed loop pendulum angle and open loop. The responses are shown in the Fig. 13 and 14 against the optimal angular trajectory generated by the LP. Fig. 13 shows the effect of keeping *l = 0.24 m*. Fig. 14 shows the effect of changing from *l = 0.24 m* to *l = 0.15 m*.

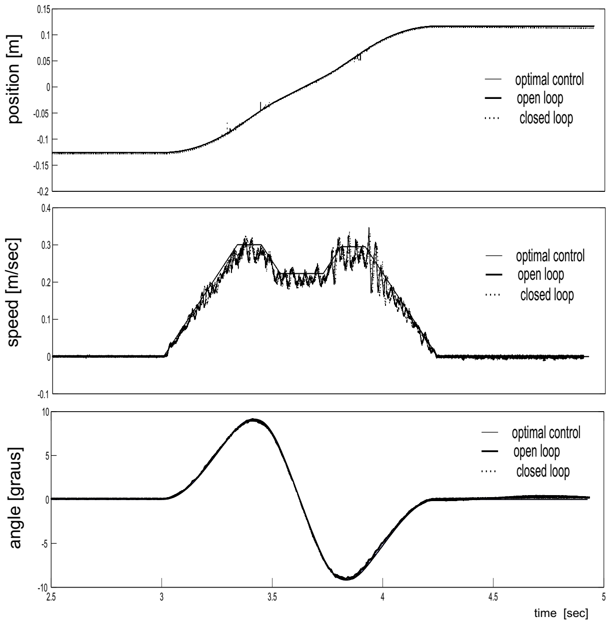

Figure 13: Cart positions, cart speed and pendulum angle for *l=0.24 m*, closed loop and open loop

In the first case, since the actual length adjusted was exactly the same used for the design, the closed-loop control does not have a noticeable influence. In the second case, shifting the mass position to approximately half of the rod course, it is noted that the angular control loop can practically restore the original behavior of the reference. This is accomplished with the expenditure of an additional control effort. That is, even though the trajectory is close to the original, there is no guarantee that minimum possible cost is achieved.

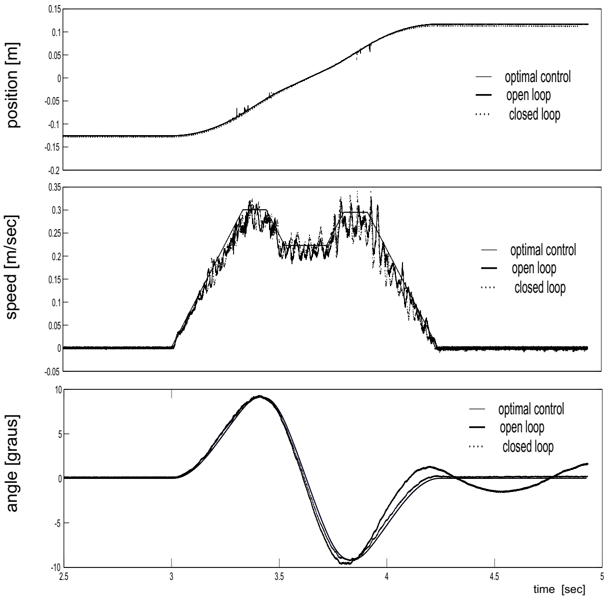

Figure 14: Cart positions, cart speed and pendulum angle for *l=0.15 m*, closed loop and open loop

4. CONCLUSIONS

This paper discusses the use of Linear Programming for the solution of a minimum time optimal control problem, including simultaneously the minimization of control effort.

The approach was applied to a cart-pendulum system and the minimal effort problem was solved explicitly by means of an objective function. Since the minimum time problem cannot be written in the form of a linear combination of the control variables, it was solved indirectly by means of a search over the sampling instants. To compensate for modeling errors and disturbances, the trajectory was maintained by means of a closed loop control system with a feed-forward action.

It was found that, by operating in open loop, the trajectory is very sensitive to modeling errors. In these tests, the modeling errors were deliberate and achieved by changing the plant dynamics by shifting the mass position attached to the rod.

The results of this study suggest that the procedure used is very suitable to cases where the real plant can be well approximated by a linear plant model. As a proposal for extending this work the following investigations are suggested.

- Use different objective functions.

- Since the problem is treated in discrete time, it seems to be reasonably simple to extend the results to time-varying systems.

- Use a model matching control structure so that the optimal control is always applied to the same plant. Thus, if the transfer function of the plant can be maintained within a certain precision, the optimal trajectory will be kept within a pre-established precision as well.

REFERENCES

Auernig, J. W., Troger, H., 1987. Time Optimal Control of Overhead Crane with Hoisting of the Load, *Automatica*, 23(4), 437–447.

Bemporad, A., Borelli, F., Morari, M., 2002. Model Predictive Control Based on Linear Programming – The Explicit Solution, *IEEE Transaction on Automatic Control*, 47(12), 1974–1985.

Blanchini, F. 1994. Ultimate Boundedness Control for Uncertain Discrete-Time Systems Via Set-Induced Lyapunov Functions. *IEEE Trans. Automatic Control*, 39(2), 428-433.

Chen, S. J., Hein, B., Wörn, H. 2007. Swing Attenuation of Suspended Objects Transported by Robot Manipulator Using Acceleration Compensation. *Proceedings of the International Conference on Intelligent Robots and Systems*, pp. 2919–2924 (San Diego, USA).

Cheng, C. C., Chen, Y. C., 1996, Controller Design for an Overhead Crane with Uncertainty, *Control Eng. Practice*, 4(5), 645–653.

Cruz, J. J., Leonardi, F., Moraes, C. C., 2008. Controle Anti-Balanço de tempo Mínimo usando Programação linear Aplicado a um Descarregador de Navios, *Anais do XVII Congresso Brasileiro de Automática*. (Juiz de Fora, Brazil).

Garrido, S., Adberrahim, M., Giménez, A., Diez, R., Balaguer, C. 2008, Anti-Swinging Input Shaping Control of an Automatic Construction Crane. *IEEE Transactions on Automation Science and Engineering*, 5(3), 549–557.

Lee, H.-H. 2004. A New Motion-Planning Scheme for Overhead Cranes With High-Speed Hoisting. *Journal of Dynamic Systems, Measurement, and Control*, 126(2), 359-365

Nassif, L. C. J., Domingos, D. F. B., Gomes, A. C. D. N. 2010. Estratégias de Controle para Reduzir Oscilações em Cargas Pendulares, *Anais do XVIII Congresso Brasileiro de Automática*. pp. 916–922 (Bonito, Brazil).

Olabi, A., Béarée, R., Gibaru, O., Damak, M. 2010. Feedrate Planning for Machining with Industrial Six-axis Robots. *Control Engineering Practice*, 471–482.

Schiehlen, W. 1997. Multibody System Dynamics: Roots and Perspectives". *Kluwer Academic Publishers*, 566–594.

Sorensen, K., Singhose, W., Dickerson, S. 2007. A Controller Enabling Precise Positioning and Sway Reduction in Bridge and Gantry Cranes. *Control Engineering Practice*, 15(7), 825–837.