FRACTIONAL POSITIVE AND STABLE TIME-VARYING CONTINUOUS-TIME LINEAR ELECTRICAL CIRCUITS

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ABSTRACT
The positivity and asymptotic stability of fractional time-varying continuous-time linear electrical circuits is addressed. Necessary and sufficient conditions for the positivity and sufficient conditions for asymptotic stability of the electrical circuits are established. New definitions of the capacitance and inductance of fractional time-varying linear electrical circuits are proposed. It is shown that there exists a large class of fractional positive and stable linear electrical circuits with time-varying parameters. Examples of positive and stable linear electrical circuits are presented.

Keywords: linear circuits, time varying circuits, positive circuits, stability

1. INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs (Farina and Rinaldi 2000, Kaczorek 2002). Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc..

The Lyapunov, Bohl and Perron exponents and stability of time-varying discrete-time linear systems have been investigated in (Czornik, Newrat, Niezabitowski, and Szyda 2012; Czornik and Niezabitowski 2013a, 2013b, 2013c; Czornik, Newrat, and Niezabitowski 2013; Czornik, Klamka, and Niezabitowski 2014). Controllability, observability and reachability of linear standard, positive and fractional electrical circuits (Kaczorek 2011a, 2011b, 2011c). The new stability tests of positive and fractional linear systems have been proposed in (Kaczorek 2011d). Fractional and positive continuous-time systems have been addressed in (Kaczorek 2008) and fractional descriptor standard and positive time-varying systems in (Kaczorek 2015a). Positivity and stability of standard and fractional descriptor time-varying discrete-time have been investigated in (Kaczorek 2015b, 2015c). Positive linear systems consisting of n subsystems with different fractional orders have been analyzed in (Kaczorek 2011e). Positivity and stability of time-varying of discrete-time and continuous-time systems have been considered in (Kaczorek 2015d, 2015e, 2015f). Stability of positive continuous-time linear systems with delays have been investigated in (Kaczorek 2009). Stability and stabilization of positive fractional linear systems by state-feedbacks have been addressed in (Kaczorek 2010).

In this paper positivity and asymptotic stability of fractional time-varying continuous-time linear electrical systems will be addressed.

The paper is organized as follows. In section 2 necessary and sufficient conditions for the positivity and sufficient conditions for the asymptotic stability of fractional time-varying continuous-time linear systems are established. The fractional positive electrical circuits with time-varying parameter are addressed in section 3. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.

2. PRELIMINARIES

Consider the fractional time-varying linear system

$$\frac{d^\alpha x}{dt^\alpha} = A(t)x \tag{1}$$

where $x = x(t) \in \Re^n$ and $A(t) \in \Re^{n\times n}$ with entries a_{ij} being continuous-time functions of time $t \in [0,+\infty)$ and

$$\frac{d^\alpha x(t)}{dt^\alpha} = {}_0 D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{dx}{d\tau} d\tau, \tag{2}$$
$$0 < \alpha < 1,$$

is the Caputo definition of $\alpha \in \Re$ order derivative of $x(t)$ and

$$\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt \tag{3}$$

is the Euler gamma function.
To find the matrix $X=X(t)\in\Re^{n\times n}$ of solutions to the equation (1) the following extension of the Picard method to fractional equations will be used

$$\frac{d^\alpha X_k}{dt^\alpha}=A(t)X_{k-1} \text{ for } k=1,2,... \tag{4}$$

with $X_0(0)=I_n$.
From (2) and (4) we obtain

$$X_k=I_n+\frac{1}{\Gamma(1+\alpha)}\int_0^t (t-\tau)^\alpha A(\tau)X_{k-1}(\tau)d\tau\,, \tag{5}$$

and $0<\alpha<1$, $k=1,2,...$.
Using (5) for $k=1,2,...$ we obtain

$$X=X(t)=I_n+\frac{1}{\Gamma(1+\alpha)}\int_0^t (t-\tau)^\alpha A(\tau)d\tau$$
$$+\frac{1}{[\Gamma(1+\alpha)]^2}\int_0^t (t-\tau)^\alpha A(\tau)\int_0^\tau (t-\tau_1)^\alpha A(\tau_1)d\tau_1 d\tau+... \tag{6}$$

Using (5) in the form

$$X_k=I_n+\int_0^t \overline{A}(\tau)X_{k-1}(\tau)d\tau \tag{7a}$$

where

$$\overline{A}(\tau)=\frac{(t-\tau)^\alpha A(\tau)}{\Gamma(1+\alpha)},\ 0\le\tau\le t \tag{7b}$$

the proof can be accomplished in a similar way as in (Gantmacher 1959, Idezak and Kamocki 2011, Kaczorek 2009) since $\overline{A}(\tau)$ is continuous and bounded function of τ. □

Definition 1. The fractional time-varying linear system (1) is called the internally positive fractional system if $x(t)\in\Re_+^n$, $t\ge 0$ for any initial conditions $x_0\in\Re_+^n$.

Theorem 1. The fractional time-varying linear system (1) is internally positive if and only if

$$A(t)\in M_n \text{ for } t\in[0,+\infty). \tag{8}$$

Proof. From (7b) it follows that $\overline{A}(t)\in M_n$ if and only if the condition (8) is satisfied. In (Kaczorek 2009) it was shown that the time-varying linear system

$$\dot{x}(t)=\overline{A}(t)x \tag{9}$$

is positive if and only if $\overline{A}(t)\in M_n$. □

In a similar way as in (Gantmacher 1959, Kaczorek 2009) the above considerations can be easily extended to the fractional time-varying linear systems

$$\frac{d^\alpha x(t)}{dt^\alpha}=A(t)x(t)+B(t)u(t) \tag{10a}$$
$$y(t)=C(t)x(t)+D(t)u(t) \tag{10b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A(t)\in\Re^{n\times n}$, $B(t)\in\Re^{n\times m}$, $C(t)\in\Re^{p\times n}$, $D(t)\in\Re^{p\times m}$ are real matrices with entries depending continuously on time $t\in[0,+\infty)$.

Definition 2. The fractional time-varying linear system (10) is called positive if $x(t)\in\Re_+^n$, $y(t)\in\Re_+^p$, $t\in[0,+\infty)$ for any initial conditions $x_0\in\Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\in[0,+\infty)$.

Theorem 2. The fractional time-varying linear system (10) is positive if and only if

$$A(t)\in M_n,\ B(t)\in\Re_+^{n\times m},$$
$$C(t)\in\Re_+^{p\times n},\ D(t)\in\Re_+^{p\times m},\ t\in[0,+\infty). \tag{11}$$

3. FRACTIONAL POSITIVE TIME-VARYING LINEAR ELECTRICAL CIRCUITS

It is well-known that for fractional time-varying electrical circuits we have

$$\frac{d^\alpha C(t)u(t)}{dt^\alpha}\ne\frac{d^\alpha C(t)}{dt^\alpha}u(t)+C(t)\frac{d^\alpha u(t)}{dt^\alpha} \tag{12a}$$

and

$$\frac{d^\alpha L(t)i(t)}{dt^\alpha}\ne\frac{d^\alpha L(t)}{dt^\alpha}i(t)+L(t)\frac{d^\alpha i(t)}{dt^\alpha} \tag{12b}$$

where $\alpha\in\Re$ is the fractional order derivative, $C(t)$ is the capacitance, $L(t)$ is the inductance, $u(t)$ is the voltage and $i(t)$ is the current.
To overcome this drawback the following definitions of the capacitance and inductance for fractional time-varying linear electrical circuits are proposed.

Definition 3. The ratio of the current $i(t)$ and α-order derivative of voltage $u(t)$, $\frac{d^\alpha u(t)}{dt^\alpha}$ i.e.

$$C(t)=\frac{i(t)}{\frac{d^\alpha u(t)}{dt^\alpha}} \tag{13}$$

is called the capacitance of fractional electrical capacitor.

Definition 4. The ratio of the voltage $u(t)$ and α-order derivative of current $i(t)$, $\frac{d^\alpha i(t)}{dt^\alpha}$ i.e.

$$L(t)=\frac{u(t)}{\frac{d^\alpha i(t)}{dt^\alpha}} \tag{14}$$

is called the inductance of fractional electrical inductor (coil).

Example 1. Consider the fractional electrical circuit shown in Figure 1 with given resistances $R_1(t)$, $R_2(t)$, $R_3(t)$, inductances $L_1(t)$, $L_2(t)$ and source voltages $e_1(t)\geq 0$, $e_2(t)\geq 0$ for $t\in[0,+\infty)$. $0<\alpha<1$

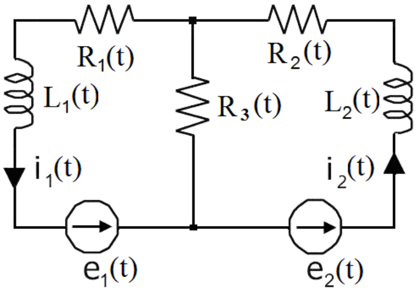

Figure 1: Electrical circuit.

Using the Definition 4 and Kirchhoff's laws we can write the equations

$$\begin{aligned} e_1(t)&=R_3(t)[i_1(t)-i_2(t)]+R_1(t)i_1(t)+L_1(t)\frac{d^\alpha i_1(t)}{dt^\alpha},\\ e_2(t)&=R_3(t)[i_2(t)-i_1(t)]+L_2(t)\frac{d^\alpha i_2(t)}{dt^\alpha}+R_2(t)i_2(t)\end{aligned} \tag{15}$$

which can be written in the form

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix} i_1(t)\\ i_2(t)\end{bmatrix}=A(t)\begin{bmatrix} i_1(t)\\ i_2(t)\end{bmatrix}+B(t)\begin{bmatrix} e(t)\\ e(t)\end{bmatrix} \tag{16a}$$

where

$$\begin{aligned} A(t)&=\begin{bmatrix} -\frac{R_1(t)+R_3(t)}{L_1(t)} & \frac{R_3(t)}{L_1(t)}\\ \frac{R_3(t)}{L_2(t)} & -\frac{R_2(t)+R_3(t)}{L_2(t)}\end{bmatrix},\\ B(t)&=\begin{bmatrix} \frac{1}{L_1(t)} & 0\\ 0 & \frac{1}{L_2(t)}\end{bmatrix}.\end{aligned} \tag{16b}$$

The electrical circuit is positive, since the matrix A is the Metzler matrix and the matrix B has nonnegative entries.

It is easy to show (Kaczorek 2015g) that in general case the linear fractional electrical circuit composed of time-varying resistors, coils and voltage sources is positive for any values of the resistances, inductances and source voltages if and only if the number of coils is less or equal to the number of its linearly independent meshes and the directions of the mesh currents are consistent with the directions of the mesh source voltages.

Example 2. Consider the fractional time-varying electrical circuit shown in Fig. 1 with given nonzero resistances $R_1(t)$, $R_2(t)$, $R_3(t)$, inductance $L(t)>0$, capacitance $C(t)>0$ and source voltage $e(t)\geq 0$ for $t\in[0,+\infty)$.

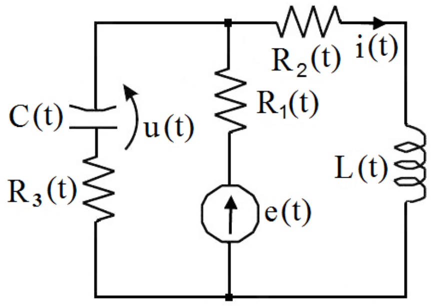

Figure 2: Fractional time-varying electrical circuit

Using the Definitions 3, 4 and Kirchhoff's laws, we can write the equation

$$\begin{aligned} e(t)&=R_1(t)\left[i(t)+C(t)\frac{d^\alpha u(t)}{dt^\alpha}\right]+R_3(t)\left[C(t)\frac{d^\alpha u(t)}{dt^\alpha}\right]\\ &+u(t),\\ e(t)&=R_1(t)\left[i(t)+C(t)\frac{d^\alpha u(t)}{dt^\alpha}\right]+L(t)\frac{d^\alpha i(t)}{dt^\alpha}\\ &+R_2(t)i(t).\end{aligned} \tag{17}$$

The equations (17) can be written in the form

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix} i(t)\\ u(t)\end{bmatrix}=A(t)\begin{bmatrix} i(t)\\ u(t)\end{bmatrix}+B(t)e(t), \tag{18a}$$

where

$$A(t)=\begin{bmatrix} 0 & [R_1(t)+R_3(t)]C(t) \\ L(t) & R_1(t)C(t) \end{bmatrix}^{-1}\begin{bmatrix} -R_1(t) & -1 \\ -R_1(t)-R_2(t) & 0 \end{bmatrix}$$
$$=\begin{bmatrix} \frac{R_1^2(t)}{[R_1(t)+R_3(t)]L(t)}-\frac{R_1(t)+R_2(t)}{L(t)} & -\frac{R_1(t)}{[R_1(t)+R_3(t)]L(t)} \\ -\frac{R_1(t)}{[R_1(t)+R_3(t)]C(t)} & -\frac{1}{[R_1(t)+R_3(t)]C(t)} \end{bmatrix},$$
$$B(t)=\begin{bmatrix} 0 & [R_1(t)+R_3(t)]C(t) \\ L(t) & R_1(t)C(t) \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$=\begin{bmatrix} \frac{R_3(t)}{[R_1(t)+R_3(t)]L(t)} \\ \frac{1}{[R_1(t)+R_3(t)]C(t)} \end{bmatrix}. \tag{18b}$$

From (18b) it follows that $A(t)\in M_2$ if and only if $R_1(t)=0$ for $t\in[0,+\infty)$. Therefore, the electrical circuit is a fractional positive time-varying system if and only if $R_1(t)=0$ for $t\in[0,+\infty)$.

Now let us consider electrical circuit shown on Fig. 3 with given positive resistances $R_k(t),\ k=0,1,\ldots,n$, inductances $L_i(t),\ i=2,4,\ldots,n_2$, capacitances $C_j(t),\ j=1,3,\ldots,n_1$ depending on time t and source voltages $e_1(t),e_2(t),\ldots,e_n(t)$. We shall show that this electrical circuit is a fractional positive time-varying linear system.

Using the Definitions 3, 4 and Kirchhoff's law we can write the equations

$$e_1(t)=R_k(t)C_k(t)\frac{d^\alpha u_k(t)}{dt^\alpha}+u_k(t) \tag{19a}$$
for $k=1,3,\ldots,n_1$,

$$e_1(t)+e_k(t)=L_k(t)\frac{d^\alpha i_k(t)}{dt^\alpha}+R_k(t)i_k(t)+u_k(t) \tag{19b}$$
for $k=2,4,\ldots,n_2$,

The equations (19) can be written in the form

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix} u(t) \\ i(t) \end{bmatrix}=A(t)\begin{bmatrix} u(t) \\ i(t) \end{bmatrix}+B(t)e(t) \tag{20a}$$

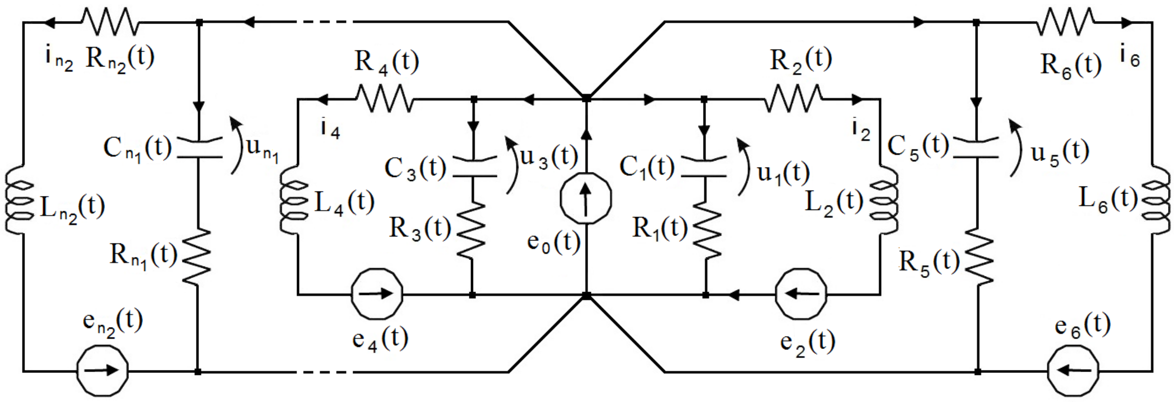

Figure 3: Fractional positive time-varying electrical circuit.

where

$$u(t)=\begin{bmatrix} u_1(t) \\ u_3(t) \\ \vdots \\ u_{n_1}(t) \end{bmatrix},\ i(t)=\begin{bmatrix} i_2(t) \\ i_4(t) \\ \vdots \\ i_{n_2}(t) \end{bmatrix},\ e(t)=\begin{bmatrix} e_1(t) \\ e_3(t) \\ \vdots \\ e_n(t) \end{bmatrix}, \tag{20b}$$
$(n=n_1+n_2)$

and

$$A(t)=\mathrm{diag}[-a_1(t),-a_3(t),\ldots,-a_{n_1}(t),-a_2(t),-a_4(t),\ldots,-a_{n_2}(t)],$$
$$a_k(t)=\frac{1}{R_k(t)C_k(t)}\ \text{ for } k=1,3,\ldots,n_1,$$
$$a_k(t)=\frac{R_k(t)}{L_k(t)}\ \text{ for } k=2,4,\ldots,n_2,$$

$$B(t)=\begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix},\ B_1(t)=\begin{bmatrix} \frac{1}{R_1(t)C_1(t)} & 0 & 0 & \ldots & 0 \\ \frac{1}{R_3(t)C_3(t)} & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \frac{1}{R_{n_1}(t)C_{n_1}(t)} & 0 & 0 & \ldots & 0 \end{bmatrix},$$
$$B_2(t)=\begin{bmatrix} \frac{1}{L_2(t)} & \frac{1}{L_2(t)} & 0 & \ldots & 0 \\ \frac{1}{L_4(t)} & 0 & \frac{1}{L_4(t)} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{L_{n_2}(t)} & 0 & 0 & \ldots & \frac{1}{L_{n_2}(t)} \end{bmatrix}. \tag{20c}$$

The electrical circuit is a fractional positive time-varying linear system since all diagonal entries of the matrix $A(t)$ are negative functions of $t \in [0,+\infty)$ and the matrix $B(t)$ has nonnegative entries for $t \in [0,+\infty)$.

4. CONCLUDING REMARKS

The positivity of fractional time-varying continuous-time linear electrical circuits have been addressed. Necessary and sufficient conditions for the positivity of the system and electrical circuits have been established. New definitions of the capacitance and inductance of fractional time-varying electrical circuits have been proposed. It has been shown that there exists a large class of fractional positive electrical circuits with time-varying parameters. The considerations have been illustrated by fractional positive electrical circuits. The consideration can be extended to a class of fractional time-varying nonlinear electrical circuits.

ACKNOWLEDGMENTS

This work was supported by National Science Centre in Poland under work No. 2014/13/B/ST7/03467.

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