A MODEL FOR IRREGULAR PHENOMENA IN URBAN TRAFFIC

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ABSTRACT
In this paper, we consider an analysis of car dynamics and its optimization on urban networks of City type, namely rectangular networks with roads of unequal length. In particular, we study the traffic variations due to changes of permeability parameters, that describe the amount of flow allowed to enter a junction from incoming roads. On each road, we distinguish a free and a congested regime, characterized by an arrival and a departure flow, respectively. Dynamics at nodes of the network is solved maximizing the through flux. The evolution on the whole network gives rise to very complicated equations, as car traffic at a single node may involve time – delayed terms from all other nodes. Hence, the network solution is found by an alternative hybrid approach, via the introduction of additional logic variables. Finally, simulations on a portion of the Salerno network, in Italy, allows to test the obtained results.

Keywords: traffic dynamics, control theory, simulation, optimization.

1. INTRODUCTION

Urban areas are often characterized by strange phenomena for car traffic: high car densities, leading to various congestion types; reductions of velocities for transport vehicles; pollutions problems, mainly due to fuel consumption. From a more specific point of view, traffic flows, especially in cities, are basic examples of material flows, mostly organized in networks.

Traffic flows have been modeling for years via several approaches (Bretti et al. 2006; Daganzo, 1995b; Helbing et al. 2005; Herty and Klar, 2003; Herty and Klar, 2004; Herty et al. 2006; Hilliges et al. 1995; Lebacque and Khoshyaran, 2005); some of them are based on conservation laws (Coclite et al. 2005; Garavello et al. 2006). The reason for such a choice is simple: the solutions of these equations have nonlinear characteristics, very useful to describe almost all the dynamic effects of car traffic, especially for vehicle queues. Although it is proved that conservation laws are a possible valid alternative for urban traffic models, road network modeling always represents a hard task, considering that the adoption of conservation laws does not always guarantee that: phenomena of daily lives are well described, such as traffic jams in some road sections (see Daganzo, 1995a; Helbing, 2001; Kerner, 2004; Schönhof et al. 2006); it is not always possible to define a total solution for the overall urban networks and, as a consequence, a global optimization procedure for traffic flows. This last problem is highly non trivial. In fact, although it often happens that traffic congestions have to be reduced in some little portions of networks (see Cascone et al. 2007; Cascone et al. 2008; D'Apice et al. 2011), the necessity of a total redesign of network roads and junctions is often required, with necessity of finding solutions of global type and extendible to various network topology. For this reason, we need a model that, beside all advantages due to conservation laws formulation, is able to focus on the overall network dynamics.

In order to achieve this aim, in this work we use a two – phase model for flows on roads (see Helbing et al. 2007; Rarità et al. 2010). In particular, the road is decomposed into road sections (links) of homogeneous capacity and nodes for their connections. Traffic dynamics along the road sections are assumed to follow the Lighthill – Whitham – Richards (briefly, LWR) model (see Lighthill et al. 1955; Richards, 1956; Whitham, 1974), but with a simplified representation, reducing the Partial Differential Equation (PDE) approach to a delayed Ordinary Differential Equation (ODE) one. For each road, two regimes are considered: free and congested. The lengths of the corresponding areas determine the exact dynamics of cars. This two – phase model, beside the obvious mathematical simplification, allows either the representation of all phenomena described via conservation laws, or the analysis of some real effects in urban traffic, such as transitions from free to congestion traffic flows due to lack of capacity, the propagation speeds of vehicles in congested traffic, spill – over effects and traffic jams, the last ones expressed by a suitable equation.

Figure 1: a portion of the real network of Salerno (up, left), consisting of two road junctions (up, right; bottom, left) and its graph (bottom, right)

Flows at nodes are regulated by permeability parameters γ, that indicate the amount of cars allowed to enter the junction from incoming roads. In Helbing et al. 2007, such parameters are assumed either zero or one, modeling the possibility of traffic lights at intersections. Here, following the approach in Rarità et al. 2010, permeabilities can also vary between zero and one. This adds the following interesting further interpretation: $0 < \gamma < 1$ indicates the situation in which the traffic is free to circulate, but only a part of it is allowed to enter the junction. This is quite normal in the usual traffic conditions, due to queues on roads that imply velocity reductions and delays in crossing the road junctions. Notice that $0 < \gamma < 1$ represents not only the possibility of modeling traffic lights, but also the normal traffic at not light – controlled road junctions.

Dynamics at nodes is, according to Coclite et al. 2005, described by the following two rules:

(A) The incoming traffic distributes to outgoing roads according to fixed (statistical) distribution coefficients;

(B) Drivers behave in order to maximize the through flux.

Here, we consider permeability parameters as controls in order to optimize the dynamics of complex networks, of "City" type, namely rectangular networks with roads of unequal length. Precisely, the variations of permeability parameters allows to establish some optimization criteria. In particular, we focus on the minimization of a cost functional, which represents the sum of queue lengths, i. e. number of delayed vehicles or lengths of congested areas. It is described that queues on roads can influence the dynamics on the whole network, leading to a "nested" equation, which cannot be solved in an analytical way.

Hence, the total solution of the network, also in terms of optimization procedures, is found using additional logic variables that represent the emptying of queues or filling the road segments. Such variables are influenced by delayed and non delayed continuous variables (queue lengths, arrival and departure flows). Indeed, they themselves influence the continuous quantities, leading to a particular system of hybrid type.

Considering the Pontryagin Maximum Principle (Bressan and Piccoli, 2007), we consider needle variations of permeability parameters, and the hybrid modeling allows either a rich description of all phenomena connected to the car traffic or the definition of a procedure to state, for the overall network, the optimal solution of minimizing car queues on roads.

The obtained results for the hybrid dynamics are tested by simulation using a modified Runge – Kutta numerical algorithm that considers delayed terms for incoming and outgoing flows into road sections. Numerical results are analyzed for a real case: a portion of Salerno urban network, which is one of the most suitable examples of rectangular network in the south of Italy. The topology of the network, represented in Figure 1, consists of three principal roads: Corso Garibaldi, Via Adolfo Cilento, Via Arturo De Felice. Road junctions are in this case of 2×2 type, namely they are characterized by two incoming roads and two outgoing roads. For this last case, it is proved (details are found in Rarità et al. 2010) that a simple needle variation of a permeability parameter provokes a wealth of variations in the other quantities at nearby nodes. This situations indicates that the hybrid approach permits, on one side, the description of the network evolution with nodes dynamics having separate equations; and, on the other hand, it keeps all characteristics of the original system.

The paper is organized as follows. Section 2 shows the model for roads, while Section 3 concerns city networks and descriptions of dynamics at nodes. In Section 4, the optimal control problem and the nested equations are analyzed; logic variables are introduced in order to define a hybrid dynamics; the variational equations, useful to define how permeability parameters influence the overall network flows, are described. Simulations for the case study are presented in Section 5. Conclusions and future perspectives are reported in Section 6.

2. FLOWS MODELLING ON ROADS

This section focuses on the car traffic behaviour on each single road of a traffic network, while the dynamics at nodes is analyzed for the special case of City type networks in next Section. The following assumptions are made: (A1) The road network consists of road sections of homogeneous capacity (links) and nodes describing their connections; (A2) A first order approach (such as LWR) gives a good description of car traffic on roads; (A3) On each road section, the fundamental diagram (density-flux graph) is well approximated by a triangular shape, with an increasing slope V_i^0 (i.e. maximum speed of vehicles, corresponding to the freed speed or speed limit on road section *i*) at low densities and a decreasing slope $c = \left(\rho_{\max} T\right)^{-1}$ in the congested regime, where: $\rho_{\max}$ denotes the maximum vehicle density in car queues; *T* is the safe time headway, which is constant along the road section; (A4) Who enters a road section first exits first (FIFO principle); (A5) Each road section has a first subsection in free phase and a second subsection in congested phase.

A road is characterized by (see Helbing et al. 2007; Rarità et al. 2010): the *arrival flow* $A_j(t)$, which indicates the inflow of vehicles into the upstream end of road section *j*; the *departure flow* $O_j(t)$, which is the flow of vehicles leaving road section *j* at its downstream end; the maximum in – or outflow of road sections *j*, $\hat{Q}_j = \left[T + \left(V_j^0 \rho_{max}\right)^{-1}\right]^{-1} = cV_j^0 \rho_{\max}\left(c + V_j^0\right)^{-1}$. All the above quantities refer to flows *per lane*, indicating by I_j the number of lanes and by L_j the length of road section *j*. Moreover, the length $l_j(t) \le L_j$ represents the length of the congested area on link *j* (measured from the downstream end), and ΔN_j is the number of stopped or delayed vehicles. An ideal representation of road section *j* is in Figure 2.

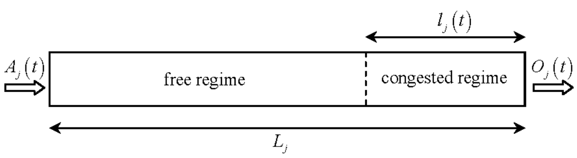

Figure 2: road section *j*

Functions $A_j(t)$ and $O_j(t)$ are also assumed upper limited by $\hat{A}_j(t)$ and $\hat{O}_j(t)$, respectively. In order to define these bounds, we refer to the following:

Definition. *For road section j, the function* $\gamma_j(t) \in [0,1]$, $t \ge 0$, *is said "permeability parameter". It defines the amount of cars that goes out from road section j.*

Remark. *Following the approach used in Rarità et al. 2010, the permeability parameter for road section j has the following interpretations:* $\gamma_j = 0$ *implies a red or amber light;* $\gamma_j = 1$ *corresponds to a green light, and all cars can flow out from road section j;* $0 < \gamma_j < 1$ *represents the green light for a situation in which not all cars can go out immediately from road section j, thus indicating that unvanished queues are still present nearby the road junction, with consequent non perfect migration of cars. Notice that* $0 < \gamma_j \le 1$ *is also useful to indicate situations in which no traffic lights are present at road intersections, but cars are free to circulate according to some yielding rules.*

We assume that $A_j(t)$ is bounded by the maximum inflow $\hat{Q}_j$, if road section *j* is not fully congested, namely $l_j(t) < L_j$; otherwise, if road section *j* is full ($l_j(t) = L_j$), $A_j(t)$ is limited by $O_j\left(t - L_j/c\right)$ a time period L_j/c before. Hence, we have $0 \le A_j(t) \le \hat{A}_j(t)$, with:

$$\hat{A}_j(t) := \begin{cases} \hat{Q}_j, & \text{if } l_j(t) < L_j, \\ O_j\left(t - L_j/c\right), & \text{if } l_j(t) = L_j. \end{cases} \tag{1}$$

Moreover, the potential departure flow $\hat{O}_j(t)$ of road section *j* is given by its permeability $\gamma_j(t)$ times the maximum outflow $\hat{Q}_j$ from this road section, if there is a queue of vehicles, namely $\Delta N_j(t) > 0$; otherwise, if road section *j* is empty ($\Delta N_j(t) = 0$), the outflow is limited by the permeability times the arrival flow $A_j\left(t - L_i/V_i^0\right)$ a time period L_i/V_i^0 before. We get that $0 \le O_j(t) \le \hat{O}_j(t)$, with:

$$\hat{O}_j(t) := \gamma_j(t) \begin{cases} A_j\left(t - L_i/V_i^0\right), & \text{if } \Delta N_j(t) = 0, \\ \hat{Q}_j, & \text{if } \Delta N_j(t) > 0. \end{cases} \tag{2}$$

2.1. An equation for traffic jams

As for traffic jams, we consider one of the suggested approaches in Helbing et al. 2007. Setting $AO^t_{j,L_j/V_j^0} := A_j\left(t - L_j/V_j^0\right) - O_j(t)$, the number of delayed vehicles for road section *j*, ΔN_j, is given by the following equation:

$$\dot{\Delta N}_j(t) = \begin{cases} AO^t_{j,L_j/V_j^0}, \; if \; t < \bar{t} \; or \\ t \geq \bar{t} \; and \; AO^t_{j,L_j/V_j^0} < 0, \\ L_j \rho_{\max}, \; if \; t \geq \bar{t} \; and \; AO^t_{j,L_j/V_j^0} \geq 0, \end{cases} \quad (3)$$

where $\bar{t}$ is the time instant for which $\Delta N_j(\bar{t}) = L_j \rho_{\max}$. It is proved in Rarità et al. 2010 that (3) represents an alternative exhaustive formulation of LWR model for roads modelling.

Remark. *Notice that dynamics of traffic queues is also considered in (2), replacing* $l_j(t) < L_j$ *by* $\Delta N_j(t) < N_j^{\max} := L_j \rho_{\max}$ *and* $l_j(t) = L_j$ *by* $\Delta N_j(t) = N_j^{\max}$. *This corresponds to a situation in which the vehicles would not queue up along the road section, but at its downstream end.*

3. CITY NETWORKS

A City network is given by a rectangular network, seen as a matrix with $\mathcal{N}$ rows and $\mathcal{M}$ columns. In particular, the network is described by the couple $(\mathcal{I}, \mathcal{J})$, where $\mathcal{I}$ and $\mathcal{J}$ indicate, respectively, the set of roads and junctions. Moreover $\mathcal{I} = \mathcal{I}_C \cup \mathcal{I}_R$, where $\mathcal{I}_C$ and $\mathcal{I}_R$ represent, respectively, the set of vertical and horizontal roads (columns and rows of the network graph). Each node is identified by a couple $(i, j) \in \mathcal{J}$, with $i \in \mathcal{N}$ and $j \in \mathcal{M}$, and has two incoming and two outgoing roads (junction of 2×2 type). At node (i, j), vertical roads are labelled as C_{ij} (entering) and C_{i+1j} (exiting), while horizontal ones are indicated by R_{ij} (entering) and R_{ij+1} (exiting), as in Figure 3.

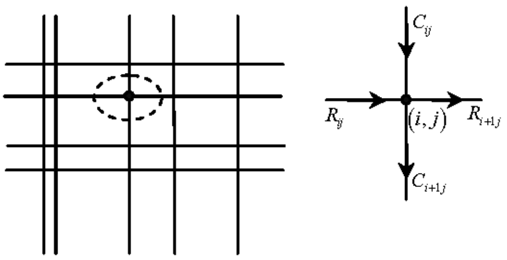

Figure 3: City network (left) and zoom on a portion (right)

To simplify the notation, we make the following assumption: **(CN)** All roads $C_{ij} \in \mathcal{I}_C$, $R_{ij} \in \mathcal{I}_R$, $i \in \mathcal{N}$, $j \in \mathcal{M}$, have the same maximum in – and outflow, i.e. $\hat{Q}_k = \hat{Q} \quad \forall \, k \in \mathcal{I}$ and free speed: $V_k^0 = V_0 \quad \forall \, k \in \mathcal{I}$. Notice, however, that we consider possibly different lengths of roads $L_{C_{ij}}$ and $L_{R_{ij}}$.

Permeability parameters of roads C_{ij} and R_{ij} are indicated by $\gamma_{C_{ij}}(t)$ and $\gamma_{R_{ij}}(t)$. As the two roads belong to the same node (i, j), we assume that $0 \leq \gamma_{C_{ij}}(t) + \gamma_{R_{ij}}(t) \leq 1$.

3.1. Traffic at nodes

The dynamics at nodes is defined by solutions to Riemann problems, i.e. Cauchy problems with initial constant data on each road. The map, which associates to every initial data the corresponding fluxes at the node, is called Riemann Solver and indicated by *RS*. The solution depends on initial fluxes and on the number of delayed vehicles (resp. length of congested zone) of all roads meeting at the node. Now, we consider two rules (see Coclite et al. 2005; Garavello and Piccoli, 2006) to define uniquely the solution to an *RS*: (A) At each node $(i, j) \in \mathcal{J}$ drivers distribute according to fixed coefficients, given by a matrix X; (B) Respecting (A), drivers behave so as to maximize the flux through node (i, j).

Remark. *Considering road junctions of* 2×2 *type, we assume that:*

$$X = \begin{pmatrix} \alpha_{ij} & \beta_{ij} \\ 1 - \alpha_{ij} & 1 - \beta_{ij} \end{pmatrix}, \quad (4)$$

where $0 \leq \alpha_{ij}, \beta_{ij} \leq 1$ *and* α_{ij} *(resp.* β_{ij}*) represents the percentage of traffic that, from road* C_{ij} *(resp.* R_{ij}*), goes to road* R_{ij+1} *(resp.* C_{ij+1}*).*

Remark. *Using both rules (A) and (B) under the assumption* $\alpha_{ij} \neq \beta_{ij}$, *we get a rich set of possible solutions for the dynamics at road junctions, depending on the state of roads, namely if they are empty, almost congested or totally congested. Details are in Rarità et al. 2010.*

From formulas (1) and (2), we also get that the 4 – tuple $\left(A_{C_{i+1j}}, A_{R_{ij+1}}, O_{C_{ij}}, O_{R_{ij}}\right)$, defined by the *RS* for the node $(i, j) \in \mathcal{J}$, is essentialy determined by: α_{ij}; β_{ij}; $\gamma_{R_{ij}}$; $\gamma_{C_{ij}}$; ΔN of roads connected to (i, j); delayed (A, O) for other nodes.

4. AN OPTIMAL CONTROL PROBLEM AND A HYBRID DYNAMIC

Now, we consider an optimal control problem for City Networks. The dynamics over the network is represented as a control system of the form:

$$\dot{x} = f(x, \gamma, \gamma_\delta), \quad (5)$$

where x is the state (the number of delayed vehicles ΔN), γ is the control (the permeability parameters)

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and γ_δ are delayed controls. Introduce the variable y_k such that $\dot{y}_k = \Delta N_k\left(x,\gamma,\gamma_\delta\right)$, $y_k\left(0\right)=0$, $i\in\mathcal{I}$. For a class Γ of admissible controls, we state the following optimal control problem:

$$\min_{\gamma\in\Gamma}\sum_{k\in\mathcal{I}} y_k\left(t\right) \tag{6}$$

for a fixed initial condition $\bar{x}$. Notice that (6) represents the minimization of delayed vehicles over the whole network in terms of permeability parameters. Now, we consider the dynamics in detail.

4.1. Queues on roads

Fix a generic node $\left(i,j\right)\in\mathcal{J}$. The dynamics for the whole network is described by the system (5), where $x=\left(y_{C_{ij}},y_{R_{ij}},\Delta N_{C_{ij}},\Delta N_{R_{ij}}\right)$.

First, assume that, for roads C_{ij} and R_{ij}, $\Delta N_{C_{ij}}>0$ and $\Delta N_{R_{ij}}>0$. Omitting, for simplicity, the dependence on traffic distribution coefficients, which are not dependent on time, we get the following equations, where the evolution of queues is function, through *RS*, of delayed and non delayed controls at node $\left(i,j\right)$:

$$\begin{aligned}
&\dot{y}_{C_{ij}}=\Delta N_{C_{ij}},\ \dot{y}_{R_{ij}}=\Delta N_{R_{ij}},\\
&\Delta\dot{N}_{C_{ij}}=RS\left(\left(\gamma_{C_{ij}},\gamma_{R_{ij}}\right)\left(t\right),\left(\gamma_{C_{i-1j}},\gamma_{R_{i-1j}}\right)\left(t-L_{C_{ij}}/V_0\right)\right),\\
&\Delta\dot{N}_{R_{ij}}=RS\left(\left(\gamma_{C_{ij}},\gamma_{R_{ij}}\right)\left(t\right),\left(\gamma_{C_{ij-1}},\gamma_{R_{ij-1}}\right)\left(t-L_{R_{ij}}/V_0\right)\right).
\end{aligned} \tag{7}$$

Consider now that roads C_{ij} and R_{ij} are empty, namely $\Delta N_{C_{ij}}=\Delta N_{R_{ij}}=0$. Dropping as usual the dependence on traffic distribution coefficients, for road C_{ij} we have:

$$\begin{aligned}
&\dot{y}_{C_{ij}}=\Delta N_{C_{ij}},\\
&\Delta\dot{N}_{C_{ij}}=g_1\left(A_{C_{ij}}\left(t-L_{C_{ij}}/V_0\right),O_{C_{ij}}\left(t\right)\right),
\end{aligned} \tag{8}$$

where g_1 is some function. Considering for simplicity the only presence of nodes $\left(i-1,j\right)$ and $\left(i-2,j\right)$ inside the network, $A_{C_{ij}}\left(t-L_{C_{ij}}/V_0\right)$ is written as:

$$\begin{aligned}
&A_{C_{ij}}\left(t-\frac{L_{C_{ij}}}{V_0}\right)=RS\left(g_2\left(RS\right),\left(O_{R_{i-1j}},\gamma_{C_{i-1j}},\gamma_{R_{i-1j}}\right)\left(t-\frac{L_{C_{ij}}}{V_0}\right)\right),\\
&g_2\left(RS\right)=g_2\left(RS\left(\left(O_{C_{i-2j}},O_{R_{i-2j}},\gamma_{C_{i-2j}},\gamma_{R_{i-2j}}\right)\left(t-\frac{L_{C_{i-1j}}+L_{C_{ij}}}{V_0}\right)\right)\right),
\end{aligned} \tag{9}$$

where g_2 is a function different from g_1. Notice that (9) represents a “nested equation”, as phenomena at $\left(i,j\right)$ are dependent on other nodes, namely the evolution of $y_{C_{ij}}$ and $\Delta N_{C_{ij}}$, expressed by (8), is written in terms of all nodes of the network. For road R_{ij}, we have similar equations.

4.2. A hybrid dynamic and needle variations

Here, we consider a hybrid dynamic to avoid the nested equations in case of empty queues. Continuous equations involving the whole network can be replaced introduced some extra logic variables. The latter, in turn, are affected and affect the continuous variables evolution.

Define the logic variables $\varepsilon_{C_{ij}}$ as:

$$\varepsilon_{C_{ij}}:=\begin{cases}-1, & \text{if } \Delta N_{C_{ij}}=0,\\ 0, & \text{if } 0<\Delta N_{C_{ij}}<\Delta N_{C_{ij}}^{\max},\\ +1, & \text{if } \Delta N_{C_{ij}}=\Delta N_{C_{ij}}^{\max}.\end{cases} \tag{10}$$

We set the following: $\tilde{\gamma}:=\left(\gamma_{C_{i-1j}},\gamma_{R_{i-1j}}\right)$, $\underset{\sim}{\gamma}:=\left(\gamma_{C_{ij}},\gamma_{R_{ij}}\right)$, $\tilde{O}:=\left(O_{C_{ij}},O_{R_{i-1j+1}}\right)$, $\underset{\sim}{O}:=\left(O_{C_{i+1j}},O_{R_{ij+1}}\right)$, $\tilde{A}:=\left(A_{C_{i-1j}},A_{R_{i-1j}}\right)$, $\underset{\sim}{A}:=\left(A_{C_{ij}},A_{R_{ij}}\right)$, $\tilde{\varepsilon}:=\left(\varepsilon_{C_{i-1j}},\varepsilon_{R_{i-1j}},\varepsilon_{C_{ij}},\varepsilon_{R_{i-1j+1}}\right)$, and $\underset{\sim}{\varepsilon}:=\left(\varepsilon_{C_{ij}},\varepsilon_{R_{ij}},\varepsilon_{C_{i+1j}},\varepsilon_{R_{ij+1}}\right)$. A complete hybrid dynamics for node $\left(i,j\right)$ is given by the following equations (for simplicity, the dependence of distribution coefficients on time is omitted, while the exponent δ indicates a delayed dependence on time):

$$\begin{aligned}
&\dot{y}_{C_{ij}}=\Delta N_{C_{ij}},\ \Delta\dot{N}_{C_{ij}}=A_{C_{ij}}^{\delta}-O_{C_{ij}},\\
&A_{C_{ij}}=RS\left(\tilde{\gamma},\tilde{A}^{\delta},\tilde{O}^{\delta},\tilde{\varepsilon}\right),\ O_{C_{ij}}=RS\left(\underset{\sim}{\gamma},\underset{\sim}{A}^{\delta},\underset{\sim}{O}^{\delta},\underset{\sim}{\varepsilon}\right).
\end{aligned} \tag{11}$$

For $\varepsilon_{R_{ij}}$, the definition is similar, substituting *C* with *R*. Moreover, also for road R_{ij} we have equations similar to (11). Suitable differences are already explained in Rarità et al. 2010.

The dynamic of control parameters γ (and distribution coefficients α or β) influence the evolution of the couple $\left(A,O\right)$ through *RS*. In turn, the values of $\left(A,O\right)$ influence themselves through *RS* and determine the continuous dynamics of ΔN. The dynamics of ΔN defines that of y and discrete changes, through ε, of the couple $\left(A,O\right)$. In Figure 4, a summarizing scheme is reported, where *c* and *d* indicate, respectively, if the dynamics is continuous or delayed.

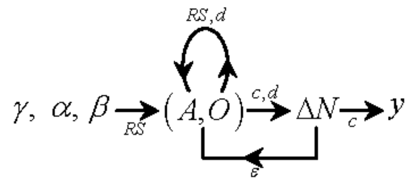

Figure 4: scheme of the hybrid dynamics

Now, we consider the sensitivity of the control system (5) with respect to control variations, adopting the point of view of Pontryagin Maximum Principle (PMP), see Bressan at al. 2007. In particular, we consider special variation of controls, called “needle variations”, and variational equations along trajectories to determine the relative effects on the dynamics.

Consider the control system (5), where $\gamma_\delta(t)=\gamma(t-\delta)$ and $\delta>0$. Fix a candidate optimal control $\Gamma \ni \gamma^*:[0,T]\mapsto U=[0,1]$ and let x^* be the corresponding trajectory, starting from a given point $\bar{x}$. A needle variation is defined as follows:

Definition *(Needle Variation). Consider the map* $\varphi: t\mapsto f\left(x^*(t),\gamma^*(t),\gamma_\delta^*(t)\right)$ *and let* τ *be a Lebesgue point for* φ. *Given* $\omega\in U$, *define a family of controls* $\eta_\gamma(t,\tau,\zeta,\omega)$, $\zeta\in[0,\tau[$ *in the following way:*

$$\eta_\gamma(t,\tau,\zeta,\omega):=\begin{cases}\gamma^*(t), & \text{if } t\in[0,\tau-\zeta[,\\ \omega, & \text{if } t\in[\tau-\zeta,\tau[,\\ \gamma^*(t), & \text{if } t\in[\tau,T].\end{cases}\qquad (12)$$

and let $\eta_x(t,\tau,\zeta,\omega)$ *be the trajectories corresponding to* η_γ *with* $\eta_x(0,\tau,\zeta,\omega)=\bar{x}$. *We call the couple* $(\eta_\gamma,\eta_x)=(\eta_\gamma,\eta_x)(\tau,\omega)$ *a needle variation of* $\left(x^*,\gamma^*,\gamma_\delta^*\right)$. *If the trajectories are uniquely determined by controls we use the simplified notation* $\eta_\gamma(\tau,\omega)$.

Given a needle variation, for every time $t\geq\tau$ it is defined a curve of points $\eta_x(t,\tau,\zeta,\omega)$ that are reached at time *t* by admissible controls $\eta_\gamma\in\Gamma$. In particular, at the final time, the points $\eta_x(T,\tau,\zeta,\omega)$ are reached. If the cost is given as in (6), for γ^* to be optimal we need that: $\nabla\left(\sum_{k\in\mathcal{I}} y_k^*(T)\right)\cdot v(T)\geq 0$, where $v(t)$ is the tangent vector to the curve $\eta_x(T,\tau,\zeta,\omega)$ at $\zeta=0$, equal to $v(t)=\left.\frac{d\eta_x(t,\tau,\zeta,\omega)}{d\zeta}\right|_{\zeta=0}$.

The vector *v*, for $t>\tau$, satisfies the variational equation $\dot{v}=D_x f\left(x^*,\gamma^*,\gamma_\delta^*\right)\cdot v$, with initial condition:

$$v(\tau)=f\left(x^*(\tau),\omega,\gamma_\delta^*(\tau)\right)-f\left(x^*(\tau),\gamma^*(\tau),\gamma_\delta^*(\tau)\right),\qquad (13)$$

that presents a jump at time $\tau+\delta$, see Rarità et al. 2010. For a City network, if a variation of $\gamma_{C_{ij},R_{ij}}$ occurs at node (i,j), we have to consider the tangent vectors $v^y_{C_{ij},R_{ij}}$ and $v^{\Delta N}_{C_{ij},R_{ij}}$ for the variables $y_{C_{ij},R_{ij}}$ and $\Delta N_{C_{ij},R_{ij}}$, respectively. Hence, the variational equations are described by $\dot{v}^y_{C_{ij}}=\dot{v}^{\Delta N}_{C_{ij}}$, $\dot{v}^y_{R_{ij}}=\dot{v}^{\Delta N}_{R_{ij}}$ and $\dot{v}^{\Delta N}_{C_{ij}}=\dot{v}^{\Delta N}_{R_{ij}}=0$.

Needle variations of permeability parameters (controls) generate other needle variations for the arrival and departure flows, which in turn provokes jumps in the variational vectors for delayed vehicles. In Table 1, we summarize an exhaustive scheme of jumps due to needle variations. Notice that column 1 shows which is the parameter (γ, *A* or *O*) for which a needle variation occurs; columns 2 indicates what are the quantities on which the needle variation provokes jumps.

Table 1: scheme of needle variations and jumps

1	**2**
$\gamma_{C_{ij}}$	$A_{C_{i+1\,j}}$, $A_{R_{ij+1}}$, $O_{C_{ij}}$
$\gamma_{R_{ij}}$	$A_{C_{i+1\,j}}$, $A_{R_{ij+1}}$, $O_{R_{ij}}$
$O_{C_{ij}}$	$A_{C_{ij}}$, $A_{R_{i-1\,j+1}}$, $O_{C_{i-1\,j}}$, $O_{R_{i-1\,j}}$ if $\Delta N_{C_{ij}}=\Delta N_{C_{ij}}^{\max}$
$O_{R_{ij}}$	$A_{C_{i+1\,j-1}}$, $A_{R_{ij}}$, $O_{C_{ij-1}}$, $O_{R_{ij-1}}$ if $\Delta N_{R_{ij}}=\Delta N_{R_{ij}}^{\max}$
$A_{C_{ij}}$	$A_{R_{i-1\,j+1}}$, $O_{C_{i-1\,j}}$, $O_{R_{i-1\,j}}$ if $\Delta N_{C_{ij}}=0$
$A_{R_{ij}}$	$A_{C_{i+1\,j-1}}$, $O_{C_{ij-1}}$, $O_{R_{ij-1}}$ if $\Delta N_{R_{ij}}=0$

The interpretations of Table 1 is the following: it is sufficient the variation of just one permeability parameter to provoke jumps in incoming and outgoing flows. Notice that some jumps occur only if roads are empty (case of incoming flows) or full (case of outgoing flows).

Remark. *For sake of space, we omit dynamics of jumps for logic variables, which is in Rarità et al. 2010.*

5. SIMULATIONS

We aim to illustrate the effect of a needle variation on a single permeability parameter for a given node in a network. For this reason we present some simulations of a real road network, proving that a unique little variation can provoke some cascade effects on incoming flows, and car queues.

We run some simulation for a City type network, which is a portion of the real network of Salerno (see Figure 1). In particular, according to the notations of Section 3, we label by (i,j) the junction between Corso Garibaldi and Via Adolfo Cilento; hence, $(i,j+1)$ indicates the intersection between Corso Garibaldi and Via Arturo De Felice. In particular, Corso

Garibaldi is identified by the three road segments R_{ij+1}, R_{ij}, and R_{ij-1}; Via Arturo De Felice by the road segments C_{ij+1}, and C_{i-1j+1}; Via Adolfo Cilento by C_{ij}, and C_{i+1j}. Figure 5 shows the topology of the considered network.

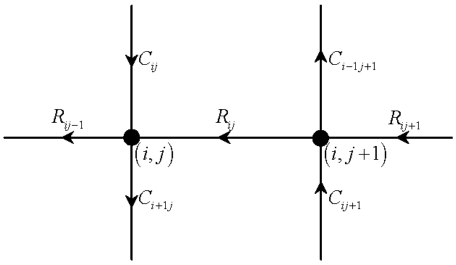

Figure 5: topology of the network

A fourth Runge – Kutta scheme is used, with temporal step $h = 0.01$ and a total simulation time $T = 25$ min. We assume that: $L_{R_{ij+1}} = 6$; $L_{C_{ij+1}} = 5$; $L_{C_{i-1j+1}} = L_{R_{ij}} = L_{C_{i+1j}} = 4$; $L_{C_{ij}} = L_{R_{ij-1}} = 3$; for all roads, $V_0 = c = 2$, $\rho_{\max} = 1$, hence $\hat{Q} = 1$; incoming fluxes:

$$A_{R_{ij+1}}(t) = A_{C_{ij+1}}(t) = A_{C_{ij}}(t) = \begin{cases} 0.5, & \text{if } t \geq 0, \\ 0, & \text{otherwise;} \end{cases} \tag{14}$$

distribution matrices $X^{(i,j)}$ and $X^{(i,j+1)}$ at nodes (i,j) and $(i,j+1)$ equal to:

$$X^{(i,j)} = \begin{pmatrix} 0.3 & 0.3 \\ 0.7 & 0.7 \end{pmatrix}; \; X^{(i,j+1)} = \begin{pmatrix} 0.2 & 0.2 \\ 0.8 & 0.8 \end{pmatrix}; \tag{15}$$

initial conditions for queues are: $\Delta N_{R_{ij+1}}(0) = 3$; $\Delta N_{C_{ij+1}}(0) = \Delta N_{C_{i-1j+1}}(0) = \Delta N_{R_{ij}}(0) = \Delta N_{C_{i+1j}}(0) = 2$, and $\Delta N_{C_{ij}}(0) = \Delta N_{R_{ij-1}}(0) = 1$; constant permeability parameters, with the exception of $\gamma_{R_{ij}}(t)$, for which a needle variation occurs, namely we have that: $\gamma_{R_{ij+1}} = \gamma_{C_{ij+1}} = \gamma_{C_{ij}} = 0.5$; $\gamma_{C_{i-1j+1}} = \gamma_{C_{i+1j}} = 0.3$, and $\gamma_{R_{ij-1}} = 0.7$; finally:

$$\gamma_{R_{ij}}(t) = \begin{cases} \gamma^*_{R_{ij}}, & \text{if } t \in [0,t_1] \cup]t_2,T], \\ \omega_{R_{ij}}, & \text{if } t \in]t_1,t_2], \end{cases} \tag{16}$$

with $\gamma^*_{R_{ij}} = 0.5$ and $\omega_{R_{ij}} = 0.2$, $t_1 = 11$ min and $t_2 = 13$ min.

Remark. *Notice that lengths of roads, velocities in free and congested regimes, initial conditions for queues and the maximal densities are normalized with respect to the length $L \simeq 116$ meters, measured on the real network that we are considering.*

In Figure 6, we present the evolution of $O_{R_{ij}}(t)$, while $A_{R_{ij}}(t)$ and $\Delta N_{R_{ij}}(t)$ are represented in Figures 7 and 8, respectively.

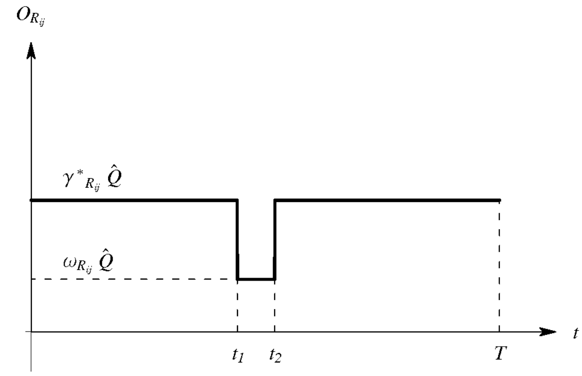

Figure 6: $O_{R_{ij}}(t)$ due to a needle variation for $\gamma_{R_{ij}}(t)$

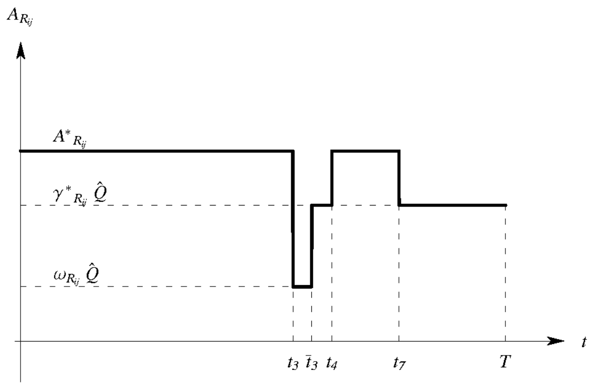

Figure 7: $A_{R_{ij}}(t)$ due to a needle variation for $\gamma_{R_{ij}}(t)$

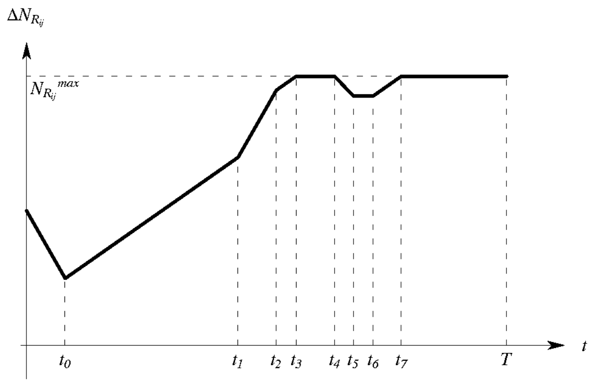

Figure 8: $\Delta N_{R_{ij}}(t)$ due to a needle variation for $\gamma_{R_{ij}}(t)$

Notice that: for $t \leq t_0 = L_{R_{ij}}/V_0 = 2$, $\Delta N_{R_{ij}}(t)$ decreases, as the solutions of *RS* at nodes $(i,j+1)$ and (i,j) imply, respectively, $A_{R_{ij}}(t - t_0) = A^*_{R_{ij}} = 0.7$ and

$O_{R_{ij}}(t)=\gamma_{R_{ij}}(t)\hat{Q}=0.5$, with $A_{R_{ij}}(t-t_0)-O_{R_{ij}}(t)>0$. At $t=t_1$, the needle variation for $\gamma_{R_{ij+1}}(t)$ generates a needle variation for $O_{R_{ij}}(t)$, that provokes an immediate change of slope for $\Delta N_{R_{ij}}(t)$. At $t=t_2$, the needle variation for $\gamma_{R_{ij+1}}(t)$ vanishes, hence we get an immediate further change of slope for $\Delta N_{R_{ij}}(t)$ while $O_{R_{ij}}(t)$ comes back to the nominal value imposed by *RS* at node (i,j). At $t_3 \simeq 14$ min, $\Delta N_{R_{ij}}(t)=\Delta N_{R_{ij}}^{\max}$ and $A_{R_{ij}}(t)$ follows the delayed $O_{R_{ij}}(t)$, namely:

$$A_{R_{ij}}(t)=O_{R_{ij}}\left(t-\frac{L_{R_{ij}}}{c}\right)=\begin{cases}\omega_{R_{ij}}\hat{Q}=0.2, \text{ if } t\in\left[t_3,\bar{t}_3\right],\\ \gamma^*_{R_{ij}}\hat{Q}=0.5, \text{ if } t\in\left]\bar{t}_3,t_4\right],\end{cases} \quad (17)$$

where $\bar{t}_3=t_2+t_0$, $t_4=t_3+t_0$. At t_4, $\Delta N_{R_{ij}}(t)$ starts to decrease as $A_{R_{ij}}(t-t_0)-O_{R_{ij}}(t)<0$, and $A_{R_{ij}}(t)=A^*_{R_{ij}}=0.7$, the value imposed by *RS* at node $(i,j+1)$; $\Delta N_{R_{ij}}(t)$ becomes constant for $t\in\left[t_5,t_6\right[$, with $t_5=\bar{t}_3+t_0$, $t_6=t_4+t_0$, as $A_{R_{ij}}(t-t_0)-O_{R_{ij}}(t)=0$. At t_6, $\Delta N_{R_{ij}}(t)$ starts to increase, and it grows until $t_7\simeq 19.5$ min, for which $\Delta N_{R_{ij}}(t)=\Delta N_{R_{ij}}^{\max}$ and, as a consequence, $A_{R_{ij}}(t)=O_{R_{ij}}\left(t-\frac{L_{R_{ij}}}{c}\right)=\gamma^*_{R_{ij}}\hat{Q}=0.5$. Moreover, $\Delta N_{R_{ij}}(t)$ remains at its maximal value, as $A_{R_{ij}}(t-t_0)-O_{R_{ij}}(t)=0$ for $t\geq t_7+t_0$.

A further analysis can also be made. For the network of Figure 5, we want to solve the optimization control problem (6). From a theoretical point of view, it is necessary to find a set of permeability parameters such as to minimize the sum of queues for all roads, namely $Y(t)=\sum_{k\in\mathcal{I}} y_k(t)$, $k\in\mathcal{I}$, where $\mathcal{I}$ is the set of roads, $\mathcal{I}\in\left\{R_{ij+1},R_{ij},R_{ij-1},C_{ij+1},C_{i-1j+1},C_{i+1j},C_{ij}\right\}$. As we introduced logic variables and defined an hybrid framework to avoid nested equations, the minimization of $Y(t)$ is simply found analyzing needle variations of the only permeability parameters, as their only variation provokes cascade effects on the whole network in terms of incoming flows, outgoing flows and queues on roads. For the portion of the real network of Salerno, traffic at nodes (i,j) and $(i,j+1)$ is regulated through $\gamma_{R_{ij+1}}(t)$, $\gamma_{C_{ij+1}}(t)$, $\gamma_{R_{ij}}(t)$, and $\gamma_{C_{i+1j}}(t)$. A suitable choice of such parameters allows to optimize the performances on the whole network in terms of delayed vehicles.

Assume, for simplicity, that $\gamma_{R_{ij+1}}(t)+\gamma_{C_{ij+1}}(t)=1$ and $\gamma_{R_{ij}}(t)+\gamma_{C_{i+1j}}(t)=1$. Then, the choice for the optimization clearly depends only on $\gamma_{R_{ij+1}}(t)$ and $\gamma_{R_{ij}}(t)$. Using the numerical software Mathematica, it is possible to use a steepest descent method in order to find the couple $\left(\gamma^*_{R_{ij+1}},\gamma^*_{R_{ij}}\right)$ that solves problem (6). In our case, with the same simulation parameters we have considered before, we get that $\left(\gamma^*_{R_{ij+1}},\gamma^*_{R_{ij}}\right)\simeq(0.415,0.318)$ in eight iterations, starting from $\left(\gamma^0_{R_{ij+1}},\gamma^0_{R_{ij}}\right)=(0.65,0.25)$. The cost functional $Y(t)$ decreases from 13 to 3.6. In Figures 9, 10 and 11, we report how $\gamma_{R_{ij+1}}(t)$ and $\gamma_{R_{ij}}(t)$ vary according to the different steps of the numerical minimization method and, finally, the cost functional, that decreases until the steady state minimum value.

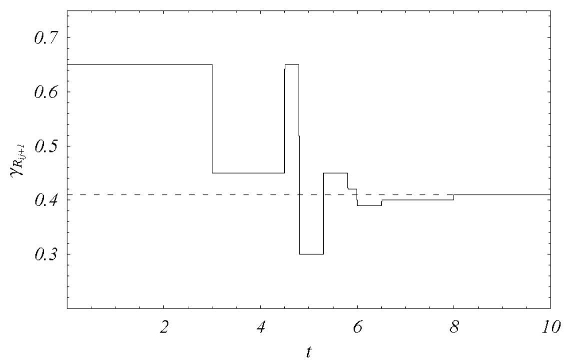

Figure 9: variations of $\gamma_{R_{ij+1}}(t)$ in different steps of the numerical minimization algorithm

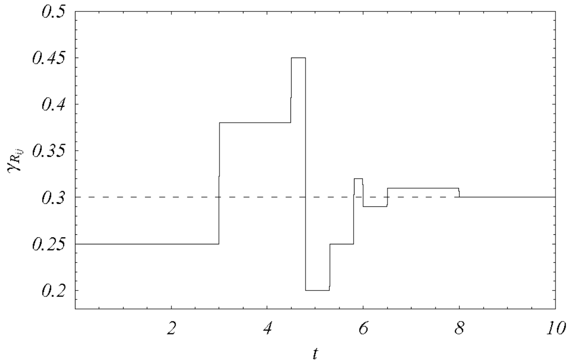

Figure 10: variations of $\gamma_{R_{ij}}(t)$ in different steps of the numerical minimization algorithm

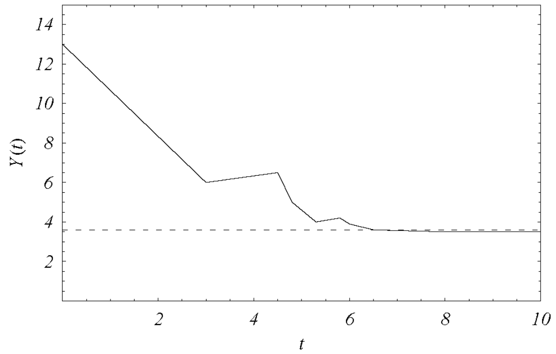

Figure 11: variations of $Y(t)$

Although it is evident that the minimization of queues is achieved, it is not possible to erase all queues on roads. A such phenomenon is not surprising, as dynamics at nodes does not always allow a complete emptying of queues, and this is the classical situation, that arises in normal traffic in cities.

6. CONCLUSIONS

We considered a delayed – ODE approach to describe car traffic in road networks of City type. The minimization of the number of vehicles was studied in terms of permeability parameters, which regulate the inflows at nodes. As the overall dynamics gives rise to nested equations, logic variables were introduced and a hybrid framework was thus obtained. A sensitivity analysis, based on needle variations, was developed for permeability parameters. The total effects of variations, also in terms of optimization of traffic performances, were described and then verified by simulations of a portion of the real network of Salerno, Italy.

Further research should be developed to achieve more information on optimal controls, e. g. using necessary conditions for hybrid control systems. This problem was not completely solved yet from a theoretical point of view.

From a numerical point of view, large scale simulations, extended to the overall network of big cities, are nowadays giving meaningful results for the optimization of traffic performances.

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