A STOCHASTIC APPROACH FOR SUPPLY SYSTEMS

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ABSTRACT

This paper focuses on a possible approach for supply systems modeled using queueing networks. According to Poisson processes, unfinished goods and control impulses arrive at the working stations, namely the nodes of the network. When the working process in a node ends, a good moves to another node with fixed probabilities either as a part to process or as a control impulse, or leaves the network. Each control impulse is activated during a random exponentially distributed time. According to some probabilities, activated impulses move an unfinished good from the node they arrive to another node, or destroy another unfinished part. For such a queueing network, a product form solution is found for the stationary state probabilities. The stability of the network, the stationary probabilities and the mean number of unfinished parts are studied via an algorithm. Such results are also useful to analyze a real system for assembling car parts.

Keywords: production systems, queueing networks, product form solution, simulation

1. INTRODUCTION

Scientific communities have always shown a great interest in modeling dynamics of industrial realities managed by supply networks and/or systems. This exigence has become deeper and deeper especially in last years, due to the growing necessity of having fast and safe processes, which could reduce, in some way, unwished phenomena, namely dead times, bottlenecks, and so on.

A great amount of mathematical approaches have been considered for this aim. Some models are continuous, mainly based on differential equations. Examples are Cutolo et al. 2011, Göttlich et al. 2006 and Pasquino et al. 2012 where, for a generic supply chain, parts dynamics is described by conservation laws, while queues, that are in front of each suppliers, are defined by ordinary differential equations. Beside continuous models, there are other ones, dealing with individual parts: some of them are based on exponential queueing networks. In this direction, a classical theoretical example is given by Jackson for waiting lines in Jackson 1957. Possible applications of queueing networks and systems are also in Yao et al. 1986, where stochastic equations are proposed for modeling supply systems, that are also analyzed in detail in Askin et al. 1993. In order to enrich the stochastic characterization of a great variety of systems, other possible variants of queueing networks have been studied. An example is given by the so called "G – networks" (see Gelenbe 1991, Gelenbe 1993), characterized by the simultaneous presence of positive customers, negative customers, signals and triggers. Positive customers are the usual ones, who join a queue in order to receive a service, and they can be destroyed by a negative customer arriving at the queue. The role of a trigger is to displace a positive customer from a queue to another one, while a signal can behave either as a negative customer or as a trigger. A vast review of G – networks is done in Artalejo 2000, Bocharov et al. 2004 and Bocharov et al. 2003, where exact solutions for queueing networks are found in "product form", which is very important as it permits the decomposition of the joint probabilities of the states of the model into products of marginal probabilities.

In this paper, considering some descriptions of G – networks in Bocharov 2002 and Gelenbe et al. 1999, we focus on a queueing network, that models a supply system, characterized either by parts dynamics or control impulses in the working stations. Unfinished parts and control signals, these last ones generated by a Central Elaboration Unit (CEU), arrive from outside the network at each node according to two independent Poisson processes. Goods are processed one by one (one server) at each node, and service times of the unfinished parts are exponentially distributed. After the working process, a good goes from a node to another one with fixed probabilities either as a part to process or a control impulse, or leaves the network. The activation time of a control impulse is exponentially distributed. Activated impulses with fixed probabilities either move a good from the node they are activated to another one or destroy an unfinished part.

For the just described queueing network, the stationary state distribution is computed in product form, and numerical results are then obtained. From simulations, we notice that the control impulses deeply influence the stationary probabilities and the mean number of parts in the network. In particular, the instability of the network easily occurs in case of low control impulses rates, although there is a high flow of unfinished goods arriving at each node.

The stability results for the queueing networks under consideration are also applied to describe a real system for assembling car parts. For such an industrial system, two separate material flows are considered: a primary flow, consisting of car skeletons, and a secondary flow for the little parts of cars, namely mirrors, glasses, wheel rims, and so on. The performances are studied via a cost functional J, that weights either the number of parts N_p, that are processed inside the system, or the amount of control signals inside nodes, N_s. A numerical analysis of J shows that it is possible to maximize N_p and to minimize N_s at the same time, with consequent advantages in terms of quality of industrial processes.

The outline of the paper is the following. Section 2 deals with the description of supply systems and its mathematical modeling. Section 3 presents the set of Chapman Kolmogorov equilibrium equations for the model. A product form solution is obtained for the steady state probabilities in Section 4. Section 5 reports some numerical results, concerning the stationary probabilities and the mean number of parts for a simple supply system and, finally, a numerical analysis of an industrial process for assembling car parts. The paper ends with Conclusions in Section 6.

2. A STOCHASTIC MODEL FOR SUPPLY SYSTEMS

We consider a supply system, which is modeled by a queueing network with the following characteristics:

- each node of the network is a working station, at which raw material flows arrive. Such flows can be either of external type, e.g. they come from outside the network, or of internal one, namely flows arrive from some inner nodes of the network;
- each node has its own working frequency, processes materials one by one, and has an infinite buffer for its own material queues;
- there exists a Central Elaboration Unit (CEU), whose aim is to give each node some electrical impulses, useful to guide dynamics in each working station;
- beside the electrical signals given by the CEU, each node has a set of non – active control impulses, that are activated if necessary. Such signals also have their own frequency action;
- if a node of the network is empty, namely there are not goods to process, the activation of a control impulse has no effect; the impulse is disabled and is not activated anymore;
- an unfinished part, once it has been processed in a given node i, either leaves the network or moves to another node j. Inside node j, the good can be further manufactured, or can behave like a control impulse. In this last case the unfinished part can destroy a good, which is inside node j, or move the good itself to another node k.

From a mathematical point of view, we deal with a queueing network with N nodes (working stations), having an infinite buffer. External arrival flows to the network are independent Poisson processes. We indicate, respectively, with a_{0i}^p and a_{0i}^c the arrival rates of external unfinished parts and electrical control signals, generated by the CEU, at node i, $i = 1,...,N$. Goods are processed one by one (one server) inside node i and the working process of a part is completed with probability $s_i^p \Delta + o(\Delta)$ in a time interval $]t, t+\Delta[$. An unfinished part, that leaves node i, moves to node j, $j = 1,...,N$, with: probability α_{ij}^p as a good that has to be processed at node j; probability α_{ij}^c as a control impulse for node j. Finally, the unfinished part leaves the network with probability $\alpha_{i0} = 1 - \sum_{j=1}^{N} \left(\alpha_{ij}^p + \alpha_{ij}^c \right)$. Indicate by $\mathbf{A^p}$ and $\mathbf{A^c}$, respectively, the matrices with elements α_{ij}^p and α_{ij}^c. The matrix $\mathbf{A} = \mathbf{A^p} + \mathbf{A^c}$, with elements $\alpha_{ij} = \alpha_{ij}^p + \alpha_{ij}^c$, is the transition matrix of a Markov chain for the dynamics of goods.

A control impulse is activated during a random time. An impulse, which is sent to node i, works in a time interval $]t, t+\Delta[$ with probability $s_i^c(c)\Delta + o(\Delta)$, provided that c non – activated control signals are present inside node i at the time instant t. When the activation period ends, a control impulse: with probability β_{ij}^p lets a good, that is inside node i, move to node j to continue the working process; with probability β_{ij}^c moves to node j an unfinished good, which belongs to node i, and the moved part behaves as a control impulse in node j. Moreover, we indicate by $\beta_{i0} = 1 - \sum_{j=1}^{N} \left(\beta_{ij}^p + \beta_{ij}^c \right)$ the probability that a control impulse destroys an unfinished good in node i. When this happens, the control impulse ends its own action and is not activated inside node i anymore. Define now the matrices $\mathbf{B^p} := \left(\beta_{ij}^p \right)$ and $\mathbf{B^c} := \left(\beta_{ij}^c \right)$. Then, the matrix $\mathbf{B} = \mathbf{B^p} + \mathbf{B^c}$, whose parameters are $\beta_{ij} = \beta_{ij}^p + \beta_{ij}^c$, is the transition matrix of a Markov chain, that describes all possible situations concerning control impulses.

The just described queueing network is identified by the couple $(\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ and $\mathcal{A}$ indicate, respectively, the set of nodes and arcs. We have that:

$\mathcal{N} = \{0,1,2,...,N\}$, where node 0 represents the external of the network, while node i, $i = 1,...,N$, is a generic working station, which belongs to the queueing network; $\mathcal{A} = \bigcup_{i \in \mathcal{N}, j \in \mathcal{N}} \{e_{ij}\}$, where e_{ij} is the arc that connects nodes i and j, from i to j. We further assume that arc e_{ij} exists if $\alpha_{ij} + \beta_{ij} > 0$, namely if some dynamics of the network involves nodes i and j. A possible graph for the queueing network is represented in Figure 1.

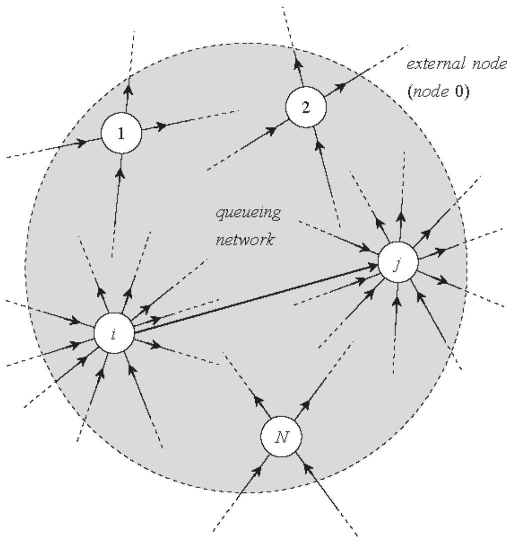

Figure 1: Possible topology for the considered queueing network

3. EQUILIBRIUM EQUATIONS

The queueing network, that models the supply system described in Section 2, is represented by a homogeneous Markov process $\{X(t), t \geq 0\}$, whose state space is $\chi = \{((p_1,c_1), (p_2,c_2),..., (p_N,c_N))\}$, with $p_i \geq 0$, $c_i \geq 0$, $i = 1,...,N$. The state $((p_1,c_1), (p_2,c_2),..., (p_N,c_N))$ has the following interpretation: at a given instant of time, there are p_1 unfinished goods and c_1 non – active impulses inside node 1, p_2 unfinished goods and c_2 non – active impulses inside node 2, and so on. Define the following quantities:

$$\mathbf{p} := (p_1, p_2,..., p_N),\ \mathbf{c} := (c_1, c_2,..., c_N),$$
$$(\mathbf{p},\mathbf{c}) := ((p_1,c_1), (p_2,c_2),..., (p_N,c_N)), \tag{1}$$

and let $\mathbf{e}_i$ be the vector, whose $i-$th component is equal to 1 while the other ones are zero. Moreover, set:

$$a_0^p := \sum_{i=1}^{N} a_{0i}^p,\ a_0^c := \sum_{i=1}^{N} a_{0i}^c. \tag{2}$$

Indicate by $\pi(\mathbf{p},\mathbf{c})$ the stationary probability of the state $(\mathbf{p},\mathbf{c})$, namely the probability that the queueing network has, for large times, p_i unfinished goods and c_i non – active impulses inside node i, $\forall\ i = 1,...,N$. If the steady state distribution $\{\pi(\mathbf{p},\mathbf{c}), \mathbf{p} \geq \mathbf{0}, \mathbf{c} \geq \mathbf{0}\}$ of the process $\{X(t), t \geq 0\}$ exists, then the following Chapman Kolmogorov equations system holds:

$$\begin{aligned}
&\pi(\mathbf{p},\mathbf{c})\left(a_0^p + a_0^c + \sum_{i=1}^{N} s_i^p \left(1-\alpha_{ii}^p\right) H(p_i) + \sum_{i=1}^{N} s_i^c(c_i) \right) = \\
&= \sum_{i=1}^{N} \pi(\mathbf{p}-\mathbf{e}_i,\mathbf{c}) a_{0i}^p H(p_i) + \sum_{i=1}^{N} \pi(\mathbf{p},\mathbf{c}-\mathbf{e}_i) a_{0i}^c H(c_i) + \\
&+ \sum_{i=1}^{N} \pi(\mathbf{p}+\mathbf{e}_i,\mathbf{c}) s_i^p \alpha_{i0} H(p_i+1) + \\
&+ \sum_{i=1}^{N} \pi(\mathbf{p}+\mathbf{e}_i,\mathbf{c}+\mathbf{e}_i) s_i^c(c_i+1) \beta_{i0} + \\
&+ \sum_{i=1}^{N} \pi(\mathbf{p},\mathbf{c}+\mathbf{e}_i) s_i^c(c_i+1)\left(1-H(p_i)\right) + \\
&+ \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \pi\left(\mathbf{p}+\mathbf{e}_i-\mathbf{e}_j,\mathbf{c}\right) s_i^p \alpha_{ij}^p H(p_i+1) H\left(p_j\right) + \\
&+ \sum_{i=1}^{N} \sum_{j=1}^{N} \pi\left(\mathbf{p}+\mathbf{e}_i,\mathbf{c}-\mathbf{e}_j\right) s_i^p \alpha_{ij}^c H(p_i+1) H\left(c_j\right) + \\
&+ \sum_{i=1}^{N} \sum_{j=1}^{N} \pi\left(\mathbf{p}+\mathbf{e}_i-\mathbf{e}_j,\mathbf{c}+\mathbf{e}_i\right) s_i^c(c_i+1) \beta_{ij}^p H\left(p_j\right) + \\
&+ \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \pi\left(\mathbf{p}+\mathbf{e}_i,\mathbf{c}+\mathbf{e}_i-\mathbf{e}_j\right) s_i^c(c_i+1) \beta_{ij}^c H\left(c_j\right) + \\
&+ \sum_{i=1}^{N} \pi(\mathbf{p}+\mathbf{e}_i,\mathbf{c}) s_i^c(c_i) \beta_{ii}^c, (\mathbf{p},\mathbf{c}) \in \chi,
\end{aligned} \tag{3}$$

where $s_i^c(0) = 0$ and $H(x)$ is a unit Heavyside function. The system (3), useful to get a mathematical expression for the steady state probability $\pi(\mathbf{p},\mathbf{c})$, has been computed considering all transitions from and to the state $(\mathbf{p},\mathbf{c})$, and balancing incoming and outgoing flows for the state $(\mathbf{p},\mathbf{c})$ (various examples of such a procedure are in Gelenbe et al. 1999).

4. STATIONARY PROBABILITIES

We want to find a general product form solution of the equations system (3), which indicates the state transitions of the presented queueing network, whose nodes have one server. With this aim, define the following quantities: $\forall\ i = 1,...,N,\quad x_i^c := a_i^c + s_i^p$,

$$\rho_i := \frac{a_i^p}{x_i^c}; \qquad q_i^c(j) := \frac{a_i^c}{s_i^c(j)},\ \forall\ i = 1,...,N,\ j = 1,...,N.$$

Notice that ρ_i represents the stationary probability that the queue of the working station i is busy. Moreover, the following traffic equations hold (see Artalejo 2000, Askin et al. 1993, Bocharov 2000, Bocharov et al. 2003, Gelenbe 1991, Gelenbe 1993, Gelenbe et al. 1999, for more details):

$$\begin{aligned} a_i^p &= a_{0i}^p + \sum_{j=1}^{N} \rho_j \left(s_j^p \alpha_{ji}^p + a_j^c \beta_{ji}^p \right),\ i = 1,\dots,N, \\ a_i^c &= a_{0i}^c + \sum_{j=1}^{N} \rho_j \left(s_j^p \alpha_{ji}^c + a_j^c \beta_{ji}^c \right),\ i = 1,\dots,N. \end{aligned} \tag{4}$$

Equations (4) are interpreted as follows: a_i^p and a_i^c are the total steady state rates of arrival of goods and control impulses, respectively, at node i. For traffic equations, we have the following:

Theorem 1 *(Solution of traffic equations). If matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are irriducible, there exists a unique solution* $\left\{ a_i^p, a_i^c \right\}$, $i = 1,\dots,N,$ *to equations (4).*

An exhaustive idea of the proof for Theorem 1 is in Gelenbe 1991 and Gelenbe 1993.

Theorem 2 *(Product form solution for stationary probabilities). If matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are irreducible and the following conditions hold:*

$$\rho_i < 1, \quad \delta_i = \sum_{c_i=0}^{+\infty} \prod_{j=1}^{c_i} q_i^c(j) < \infty, \quad i = 1,\dots,N, \tag{5}$$

then the Markov process $\left\{ X(t), t \ge 0 \right\}$ *is ergodic and its stationary distribution is represented in product form as:*

$$\pi(\mathbf{p},\mathbf{c}) = \prod_{i=1}^{N} \pi_i(p_i, c_i), \tag{6}$$

where, $\forall \ i = 1,\dots,N,$

$$\pi_i(p_i, c_i) = (1-\rho_i)\rho_i^{p_i} \delta_i^{-1} \prod_{j=1}^{c_i} q_i^c(j),\ p_i \ge 0,\ c_i \ge 0, \tag{7}$$

and $\prod_{j=1}^{0} \equiv 1.$

Proof. The proof is based on verifying that (6) is a solution of (3). In particular, substituting the expressions of ρ_i and $q_i^c(j)$ and formulas (6) and (7) into the equilibrium equations system (3), we obtain:

$$\begin{aligned} & a_0^p + a_0^c + \sum_{i=1}^{N} s_i^p H(p_i) + \sum_{i=1}^{N} s_i^c(c_i) = \\ & = \sum_{i=1}^{N} \frac{a_{0i}^p}{\rho_i} H(p_i) + \sum_{i=1}^{N} \frac{s_i^c(c_i)}{a_i^c} a_{0i}^c + \sum_{i=1}^{N} \rho_i s_i^p \alpha_{i0} + \\ & + \sum_{i=1}^{N} \rho_i a_i^c \beta_{i0} + \sum_{i=1}^{N} a_i^c \left(1 - H(p_i)\right) + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\rho_i}{\rho_j} s_i^p \alpha_{ij}^p H(p_j) + \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N} \rho_i \frac{s_j^c(c_j)}{a_j^c} s_i^p \alpha_{ij}^c + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\rho_i}{\rho_j} a_i^c \beta_{ij}^p H(p_j) + \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N} \rho_i \frac{s_j^c(c_j)}{a_j^c} a_i^c \beta_{ij}^c. \end{aligned} \tag{8}$$

Using some simplifications, we get that:

$$\begin{aligned} & \sum_{i=1}^{N} \frac{s_i^c(c_i)}{a_i^c} a_{0i}^c + \sum_{i=1}^{N}\sum_{j=1}^{N} \rho_i \frac{s_j^c(c_j)}{a_j^c} s_i^p \alpha_{ij}^c + \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N} \rho_i \frac{s_j^c(c_j)}{a_j^c} a_i^c \beta_{ij}^c = \sum_{i=1}^{N} s_i^c(c_i), \end{aligned} \tag{9}$$

and:

$$\begin{aligned} & \sum_{i=1}^{N} \frac{a_{0i}^p}{\rho_i} H(p_i) + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\rho_i}{\rho_j} s_i^p \alpha_{ij}^p H(p_j) + \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\rho_i}{\rho_j} a_i^c \beta_{ij}^p H(p_j) = \sum_{i=1}^{N} \left(a_i^c + s_i^p\right) H(p_i). \end{aligned} \tag{10}$$

Then, from expressions (9) and (10), the equality (8) becomes:

$$\begin{aligned} & a_0^p + a_0^c + \sum_{i=1}^{N} s_i^p H(p_i) + \sum_{i=1}^{N} s_i^c(c_i) = \\ & = \sum_{i=1}^{N} s_i^c(c_i) + \sum_{i=1}^{N} \left(a_i^c + s_i^p\right) H(pi) + \sum_{i=1}^{N} \rho_i s_i^p \alpha_{i0} + \\ & + \sum_{i=1}^{N} \rho_i a_i^c \beta_{i0} + \sum_{i=1}^{N} a_i^c \left(1 - H(p_i)\right) = \\ & = \sum_{i=1}^{N} s_i^c(c_i) + \sum_{i=1}^{N} s_i^p H(pi) + \sum_{i=1}^{N} \rho_i s_i^p \alpha_{i0} + \sum_{i=1}^{N} \rho_i a_i^c \beta_{i0} + \\ & + \sum_{i=1}^{N} a_i^c = a_0^p + a_0^c + \sum_{i=1}^{N} s_i^p H(p_i) + \sum_{i=1}^{N} s_i^c(c_i), \end{aligned} \tag{11}$$

hence we have just proved an identity. Under the theorem assumptions, the process $\left\{ X(t), t \ge 0 \right\}$ is irreducible. Therefore, according to Foster's theorem (see Bocharov et al. 2004), the process is ergodic, and formulas (6) and (7) give its unique stationary distribution. This completes the proof.

5. SIMULATIONS

In this section, we examine two different simulation cases. In the first case, a general supply system is considered, for which the mean number of parts to process is computed and an analysis of stability conditions for nodes is made. In the second case, a real network for car parts is studied. Such last situation indicates that, although some instabilities can arise inside the nodes of the network, it is possible to optimize the performances of supply systems via a suitable cost functional.

5.1. A general supply system

We present some numerical results for a supply system, which is represented in Figure 2: there are five working stations (nodes). External flows of goods arrive at each node, while the CEU sends electrical impulses only at nodes 1 and 2. According to some fixed probabilities, unfinished parts can travel from node i to node $i+1$, $i=1,2,3,4$; from node 5, goods either leave the network or come back to node 1. For control impulses, the dynamics is the same of the unfinished parts.

For the just described supply system, we will consider some numerical results for the stationary probabilities and the mean number of parts in the network.

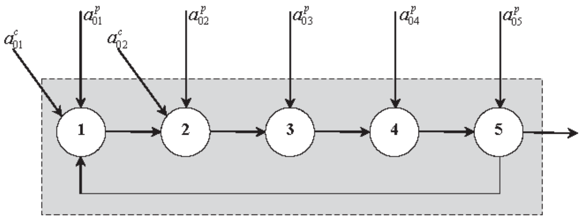

Figure 2: Scheme of the supply system

5.1.1. Numerical results

In what follows we consider some results for the queueing network of Figure 2. Assume that $a_{0i}^p = 10 \ \forall$ $i=1,...,5$, $s_1^p = 20$, $s_2^p = 40$, $s_3^p = s_4^p = 25$, $s_5^p = 30$, where all above quantities are intended as number of goods per minute; $a_{01}^c = a_{02}^c = 5$, $a_{03}^c = a_{04}^c = a_{05}^c = 0$, $s_1^c = s_2^c = s_3^c = s_4^c = 25$, $s_5^c = 30$, where a_{0i}^c and s_i^c, $i=1,...,5$, are measured as number of control impulses per minute;

$$\mathbf{A}^p = \mathbf{B}^p = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0.2 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{12}$$

$$\mathbf{A}^c = \mathbf{B}^c = \begin{pmatrix} 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

In Tables 1 and 2, we summarize some values of the stationary probability $\pi_i(p_i, c_i)$ for node i, $i=1,2$. We choose to analyze only the behaviour of nodes 1 and 2, as they are the only ones to be interested by external goods and control impulses rates. Notice that, although such rates are the same for both nodes, if the number of control signals increases, $\pi_i(p_i, c_i)$, $i=1,2$, decreases. This is not surprising, as controls in nodes provoke a variation of the ordinary goods dynamics, either in terms of movements to other nodes or destruction of parts.

Table 1: $\pi_1(p_1, c_1)$ for different values of p_1 (columns) and c_1 (rows)

$p_1 \backslash c_1$	1	2	3
1	0.0394809	0.00789617	0.00157923
2	0.0219893	0.00439787	0.00087957
3	0.0122472	0.00244944	0.00048989

Table 2: $\pi_2(p_2, c_2)$ for different values of p_2 (columns) and c_2 (rows)

$p_2 \backslash c_2$	1	2	3
1	0.0548667	0.0262527	0.0125614
2	0.0179102	0.0085697	0.0041004
3	0.0058464	0.0027974	0.0013385

In order to understand better how stationary probabilities depend on the number of goods, we define the probability $\widetilde{\pi_i}(p_i)$ that a certain node i, $i=1,...,5$, has p_i goods, namely:

$$\widetilde{\pi_i}(p_i) := \sum_{c_i=0}^{+\infty} \pi_i(p_i, c_i), \ i=1,...,5. \tag{14}$$

In Table 3, we collect some values of $\widetilde{\pi_i}$, $i=1,2$.

Table 3: $\widetilde{\pi_i}$ for node i (columns), $i=1,2$, assuming p_j unfinished goods (rows), $j=1,2,3$

$i \backslash p_j$	1	2	3
1	0.246755	0.137433	0.0765452
2	0.219874	0.0717737	0.0234292

Notice that $\widetilde{\pi_i}(p_i)$ increases when the number of goods decreases and, moreover, $\widetilde{\pi_1}(p_1) > \widetilde{\pi_2}(p_2)$, indicating that node 1 tends to have more parts than node 2. This is an evident influence of the possibility to reprocess some goods, coming from node 5, inside node 1.

Further studies can be done considering the mean number of parts in the network, namely:

$$N_p := \sum_{p_i=0}^{+\infty} p_i \left(\sum_{c_j=0}^{+\infty} c_j \pi_i(p_i, c_j) \right). \tag{15}$$

If we sketch N_p vs a_{01}^p (Figure 3, top) and vs a_{02}^p (Figure 3, bottom), we have a precise idea of the ergodicity condition of the network process. In particular, if the network is simulated with:

- a_{01}^{p} variable and other parameters equal to the ones used before, node 1 becomes instable when $a_{01}^{p} \simeq 24.88 = a_{01}^{p,*}$, leading to the instability of the overall network;
- a_{02}^{p} variable and other parameters equal to the ones used before, the network process is not ergodic anymore if $a_{02}^{p} \geq 48.89 = a_{02}^{p,*}$.

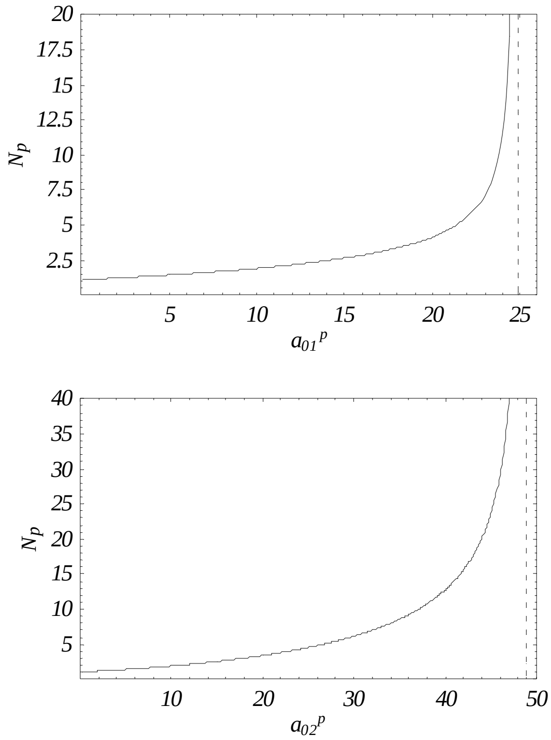

Figure 3: N_p vs a_{01}^{p} (top) and a_{02}^{p} (bottom)

A similar phenomenon happens considering the behaviour of N_p vs a_{01}^{c} (Figure 4, top) and vs a_{02}^{c} (Figure 4, bottom). We get that if:

- a_{01}^{c} is variable and the other parameters are equal to the ones used before, the condition of instability for node 1, and hence for the overall network, is achieved for $a_{01}^{c} \simeq 24.99 = a_{01}^{c,*}$;
- a_{01}^{c} varies while the other parameters are the same ones used before, node 2 is instable for $a_{02}^{c} \geq 18 = a_{02}^{c,*}$, and the network process is not ergodic anymore.

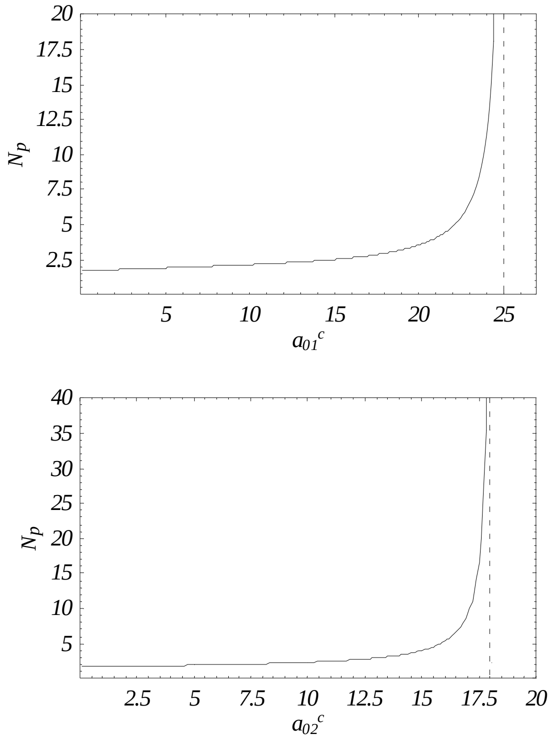

Figure 4: N_p vs a_{01}^{c} (top) and a_{02}^{c} (bottom)

Moreover, notice that, in analogy with the usual exponential queueing systems with one server, the shape of N_p in Figures 3 and 4 is the one of a hyperbolic function. In Figure 5, we represent: on the top, N_p as function of a_{01}^{p} and a_{02}^{p}; on the bottom, N_p vs a_{01}^{c} and a_{02}^{c}. In both cases, the other parameters, which are not assumed variable, are equal to the ones used for computing the stationary probabilities.

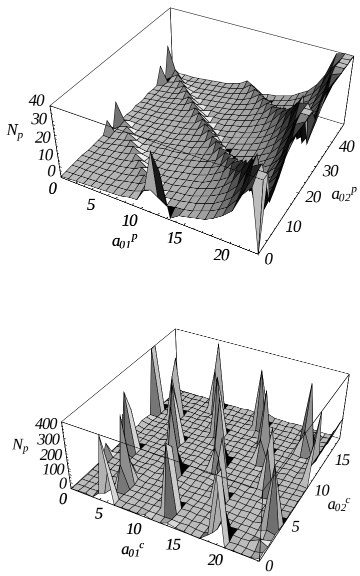

Figure 5: N_p vs a_{01}^{p} and a_{02}^{p} (top), and vs a_{01}^{c} and a_{02}^{c} (bottom)

Notice that: for Figure 5, top, N_p tends to infinity only if $a_{01}^p \simeq a_{01}^{p,*}$ and $a_{02}^p \simeq a_{02}^{p,*}$; for Figure 5, bottom, N_p approaches the infinity for various combinations of a_{01}^c and a_{02}^c, and not only for the critical values $a_{01}^{c,*}$ and $a_{02}^{c,*}$. Such effect indicates that the ergodicity of the network process is mainly influenced by control impulses rates, and this is not unusual, as controls always tend to create some natural discontinuities in the normal working processes of goods.

5.2. A real network for car parts

We describe some simulation results for the network in Figure 6, that represents a scheme of real industrial processes, that are commonly used for assembling car parts.

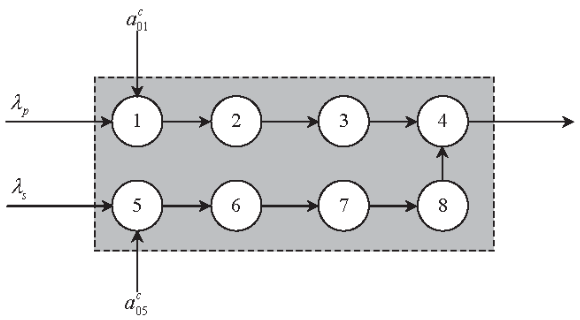

Figure 6: A network for assembling car parts

There are eight nodes and two external flows of goods. In particular, we distinguish: a primary flow, that has rate λ_p and consists of car skeletons; a secondary flow, with rate λ_s for little parts of cars, namely mirrors, glasses, wheel rims, and so on. At each node a precise activity is associated. As for car skeletons, in node 1 they are washed, dried in node 2 and then painted in node 3. For the secondary flow, instead, we have that little components of cars are washed, dried and painted in nodes 5, 6 and 7, respectively. Then, such parts are completely assembled in node 8, and a complete car is obtained in node 4. The just assembled cars go out of the network from node 4. Figure 7 sums up the complete assembling process. For such a system, we will consider some numerical results for a cost functional, that represents, using an opportune weight, the joint effect of the mean number of parts and controls inside the network.

5.2.1. Numerical results

Assume that primary and secondary flows have variable rates, respectively, $a_{01}^p = \lambda_p \in \left]0,30\right[$ and $a_{05}^p = \lambda_s \in \left]0,30\right[$. Moreover, we have: $a_{0i}^p = 0$, $i = 2,3,4,6,7,8$; $a_{01}^c = a_{05}^c = 2$, $a_{0i}^c = 0$, $i = 2,3,4$; $s_i^p = 10 \quad \forall \quad i = 1,...,4$, $s_i^p = 20 \quad \forall \quad i = 5,...,8$, $s_i^c = 1 \quad \forall \quad i = 1,...,8$, where all above quantities are measured per minute; $\mathbf{A}^c = \mathbf{B}^p = \mathbf{B}^c = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix of order 8; and $\mathbf{A}^p$ has elements:

$$\alpha_{ij}^p = \begin{cases} 1, & \text{if } j = i+1,\ i \in \{1,2,3,5,6,7\}, \\ & \text{or } i = 2j, \text{ with } j = 4, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

Notice that matrices indicate that all goods always travel as a "parts to process" from one station to the following one.

In order to describe the performances of the system, we define the following cost functional:

$$J\left(\lambda_p, \lambda_s\right) := wN_p - (1-w)N_s, \tag{17}$$

where N_p is defined as in (15), N_s is the mean number of control signals inside the network, given by:

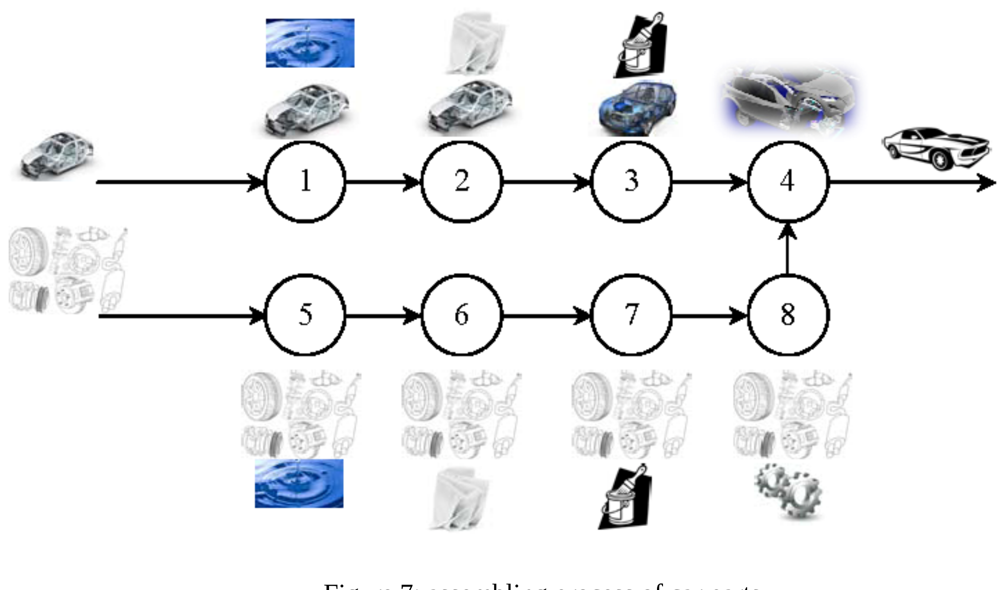

Figure 7: assembling process of car parts

$$N_s := \sum_{c_j=0}^{+\infty} c_j \left(\sum_{p_i=0}^{+\infty} p_i \pi_i \left(p_i, c_j \right) \right), \quad (18)$$

and $w \in]0,1[$ is a real number, that weights either the contribution of N_p or the one of N_s. The aim is to maximize J with respect to the couple $\left(\lambda_p, \lambda_s\right)$, namely we want to find the values of primary and secondary flows in order to: increase the mean number of parts inside the network, with consequent advantages in terms of the production itself; reduce the possibility of controlling nodes by signals. This aim is highly non-trivial as the ergodicity condition of the network process has also to be considered. Mathematically speaking, the problem is the following:

$$\begin{aligned} &\max_{\left(\lambda_p, \lambda_s\right)} J\left(\lambda_p, \lambda_s\right), \\ &\left(\lambda_p, \lambda_s\right) \in]0,30[\times]0,30[, \\ &\left(\lambda_p, \lambda_s\right) \text{ such that: } \rho_i < 1, \sum_{c_i=0}^{+\infty} \prod_{j=1}^{c_i} q_i^c\left(j\right) < \infty, \end{aligned} \quad (19)$$

where the last constraint of problem (19) indicates that $\left(\lambda_p, \lambda_s\right)$ has to be chosen in order to respect the stability condition for each node of the network.

As an analytical analysis of J is very complex, some numerical estimations have been made using the software Mathematica. For $w = \frac{1}{2}$, we have obtained that, if $\left(\lambda_p, \lambda_s\right) \in \left]\overline{\lambda}_p, 30\right[\times \left]\overline{\lambda}_s, 30\right[$, with $\overline{\lambda}_p = 22.3$ and $\overline{\lambda}_s = 24.7$, the network process is not ergodic as J tends to infinity. Indeed, for values of $\left(\lambda_p, \lambda_s\right)$ such that the network is stable, there exists a unique maximum point at $\left(\lambda_p^*, \lambda_s^*\right) \simeq \left(17.5, 16.5\right)$ for which $J\left(\lambda_p^*, \lambda_s^*\right) \simeq 6.4$, see Figure 8. Hence, the output for the car parts of the system is optimized for values of primary and secondary flows, that approach 30, the maximal possible rate.

Notice that, for other values of w, $\overline{\lambda}_p$ and $\overline{\lambda}_s$ are obviously different but $\max_{w \in]0,1[} \overline{\lambda}_p = 23.6$ and $\max_{w \in]0,1[} \overline{\lambda}_s = 25.8$, namely there is no meaningful difference with the case $w = \frac{1}{2}$. The same happens with the maximum point, for which minimal variations occur.

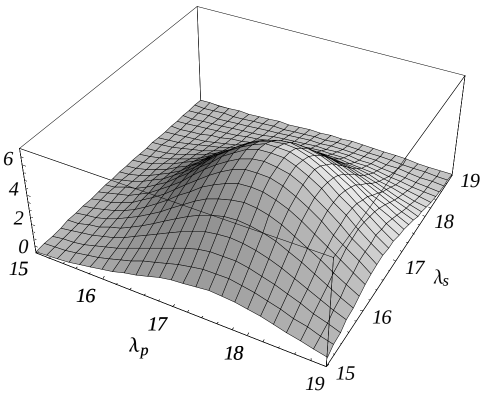

Figure 8: J vs λ_p and λ_s for $w = \frac{1}{2}$

6. CONCLUSIONS

In this paper, it has been described an exponential queueing network, which models a supply system, whose dynamics is determined either by unfinished parts or control impulses.

Steady state probabilities for such a queueing network have been found in product form.

A numerical analysis of the model has allowed to establish that the stationary probabilities are deeply influenced by control impulses, that also have a strong impact on the overall dynamics of the queueing network.

A real network for assembling car parts has been studied through a cost functional in order to maximize the mean number of parts inside the system with the minimal number of control signals.

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