TRAFFIC LIGHT SIMULATION WITH TIME-VARYING TRAFFIC DISTRIBUTION AT JUNCTIONS

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ABSTRACT
The aim of this paper is to simulate the effect of traffic lights and time-varying flows distribution at junctions of urban traffic networks. We consider a macroscopic model for road networks based on conservation law, describing the motion of cars as a continuous flow. At junctions some Riemann solvers to find a unique solution to Riemann problems are introduced.
In particular we propose a micro-algorithm to define a Riemann solver in situations in which a road in some time-instant is empty and the corresponding problem can be under-determined. Then, we discuss the correct use of Riemann solvers to capture the presence of traffic lights and time-varying behavior of drivers at junctions.
Simulation results for a 2×2 junction and a comparison among the effects of changing traffic lights cycles in a network are shown.

Keywords: fluid-dynamic model for traffic networks, conservation law, Riemann solvers, traffic lights simulation.

1. INTRODUCTION

To study car traffic phenomena, researchers from various areas proposed a lot of models, among which fluid-dynamic ones. The latter treat traffic from a macroscopic point of view: the evolution of macroscopic variables, such as density and average velocity of cars, is considered.

The basic fluid-dynamic model is due to Lighthill, Whitham and Richards (LWR model) (Lighthill and Whitham, 1955; Richards 1956), according to which the motion of cars along a road can be modeled by a conservation law, regarding the density of cars as the main quantity to be looked at. To overcome the limitations of the LWR model, other alternatives were searched for, such as second and third order models (Helbing 2001; Colombo 2002; Bellomo and Coscia 2005). Recently, the LWR model was extended to networks (Coclite, Garavello and Piccoli 2005; Garavello and Piccoli 2006).

Since traffic networks consist of a finite set of roads meeting at some junctions, the dynamics at junctions is captured solving Riemann problems which are Cauchy problems with constant initial data on each road. In order to solve uniquely Riemann problems some assumptions are made:

- the incoming traffic distributes to outgoing ones according to fixed (statistical) coefficients;
- drivers behave to maximize the through flux.

More precisely, if the number of incoming roads is greater than that of outgoing ones, some right of way parameters have to be added.

Once the solution to a Riemann problem is provided, piecewise constant approximations via a wave front tracking algorithm can be constructed (Bressan 2000; Garavello and Piccoli 2006).

In this paper starting from Coclite, Garavello, Piccoli model, we describe the evolution of vehicles flows respecting traffic lights cycles and defining time-varying distribution at junctions in order to take account the dynamic behavior of drivers. Then we introduce a micro-algorithm and a Riemann solver to cover these typical situations. Numerical schemes such as the Godunov method, based on exact solutions to Riemann problems (Godlewski and Raviart 1991; Godunov 1959) are used to solve numerically the conservation law along roads.

The paper is organized as follows. A model for traffic networks is introduced in Section 2. Section 3 is devoted to the definition of Riemann solvers at junctions. In particular the micro-algorithm and Riemann solvers for traffic lights and time-varying distribution coefficients are described. Numerical methods are presented in Section 4. Some simulation results for a typical scenario of a simple 2×2 junction (two incoming roads and two outgoing ones) are shown in order to test and verify the adopted approach in Section 5. The section ends with a comparison between different configurations of two traffic lights in a network with three junctions.

2. MODELING CAR TRAFFIC NETWORKS

A road network is schematized by the couple (I,J), where $\mathrm{I}=\{I_i : i=1,..,N\}$ represents the set of roads, while J is the collection of junctions connecting roads. Fixed a junction $J\in\mathrm{J}$, we denote by $Inc(J)$ and $Out(J)$, respectively, the set of all incoming roads, numbered from 1 to n, and the set of all outgoing ones, numbered from $n+1$ to $n+m$ (see Figure 1).

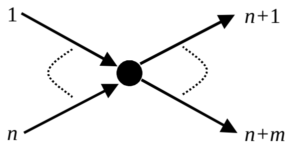

Figure 1: example of a $n \times m$ junction.

Each road is represented by an interval $I_i = [a_i, b_i] \subseteq \Re$.

According to the LWR model, we describe the evolution of cars density along each road by

$$\partial_t \rho_i + \partial_x f(\rho_i) = 0, \tag{1}$$

where $\rho_i = \rho_i(t,x) \in [0, \rho_{\max}]$ is the cars density on road I_i, $\rho_{\max}$ is the maximal density, $f(\rho) = \rho v(\rho)$ is the flux, and $v(\rho)$ is the average velocity.

On the flux f we assume that

(F) $f:[0,1] \to \Re$ is smooth, strictly concave, $f(0) = f(1) = 0$. Therefore there exists a unique strict maximum $\sigma \in]0,1[$.

The dynamics at each junction $J \in \mathrm{J}$ is determined by solving a Riemann Problem (*RP*), which is a Cauchy problem with constant initial data on each incident road. The solution is formed either by continuous waves, called *rarefactions*, or by traveling discontinuities, called *shocks*. In order to find a unique solution some Riemann Solvers (*RS*) are defined, based on rights of way and traffic distribution parameters.

Definition 1. A Riemann Solver for the junction J is a map $RS:[0,1]^n \times [0,1]^m \to [0,1]^n \times [0,1]^m$ that associates to Riemann data $\rho_0 = (\rho_{1,0}, \ldots, \rho_{n+m,0})$ at J a vector $\hat{\rho} = (\hat{\rho}_1, \ldots, \hat{\rho}_{n+m})$ so that the solution on an incoming road I_i, $i = 1, \ldots, n$, is given by the wave $(\rho_{i,0}, \hat{\rho}_i)$ and on an outgoing road I_j, $j = n+1, \ldots, n+m$, is given by the wave $(\hat{\rho}_j, \rho_{j,0})$. We require the consistency condition

(CC) $RS(RS(\rho_0)) = RS(\rho_0)$.

In particular, for a $n \times m$ junction, *RSs* are based on the following rules:

(A) drivers distribute at a junction according to some traffic distribution coefficients which represent the preferences of drivers from the incoming roads to outgoing ones and they are collected in the matrix:

$$A = \{\alpha_{ji}\}_{j=n+1,\ldots,n+m, i=1,\ldots,n} \in \Re^{m \times n} \tag{2}$$

such that $0 < \alpha_{ji} < 1$, $\sum_{j=n+1}^{n+m} \alpha_{ji} = 1$, where α_{ji} is the percentage of drivers who, arriving from the *i*th incoming road, take the *j*th outgoing road.

(B) Respecting (A) rule, drivers behave so as to maximize the flux through the junction $J \in \mathrm{J}$.

(C) If $n > m$, it is assumed that not all cars can enter the outgoing roads and let Q be the amount that can do it. Then $p_i Q$ cars come from the *i*th road to the generic *j*th one, with $i = 1, \ldots, n$ and $j = n+1, \ldots, n+m$, and p_i can be thought as a right of way parameter.

3. RIEMANN PROBLEMS AT JUNCTIONS

Let J be a $n \times m$ junction. The density functions on incoming and outgoing roads are denoted as $(t,x) \in \Re^+ \times I_i \to \rho_i(t,x) \in [0, \rho_{\max}]$, $i = 1, \ldots, n$ and $(t,x) \in \Re^+ \times I_j \to \rho_j(t,x) \in [0, \rho_{\max}]$, $j = n+1, \ldots, n+m$. We observe that the waves generated on incoming roads must have negative velocity, while the outgoing ones positive velocity. For this reason, some bounds on possible states reached by a solution to an *RP* at J exist. Precisely, if we set $\gamma_i = f(\rho_i)$, we have:

Proposition 2. Let $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ be the initial densities of an *RP* at J. The maximal fluxes $\gamma_i^{\max}$, $i = 1, \ldots, n$ and $\gamma_j^{\max}$, $j = n+1, \ldots, n+m$, that can be obtained on incoming roads and outgoing ones, respectively, are the following:

$$\gamma_i^{\max} = \begin{cases} f(\rho_{i,0}) & if\ \rho_{i,0} \in [0, \sigma], \\ f(\sigma) & if\ \rho_{i,0} \in]\sigma, \rho_{\max}], \end{cases} \quad i = 1, \ldots, n, \tag{3}$$

$$\gamma_j^{\max} = \begin{cases} f(\sigma) & if\ \rho_{j,0} \in [0, \sigma], \\ f(\rho_{j,0}) & if\ \rho_{j,0} \in]\sigma, \rho_{\max}], \end{cases} \quad j = n+1, \ldots, n+m. \tag{4}$$

Theorem 3. Let J be a $n \times m$ junction. For every initial data $(\rho_{1,0}, \ldots, \rho_{n+m,0})$, there exists an unique admissible weak solution $\rho = (\rho_1, \ldots, \rho_{m+n})$ to (1) at J, respecting rules (A), (B) and (C), such that

$$\rho_1(0, \cdot) \equiv \rho_{1,0}, \ldots, \rho_{n+m}(0, \cdot) \equiv \rho_{n+m,0}. \tag{5}$$

Moreover, there exists a unique vector $(\hat{\rho}_1, \ldots, \hat{\rho}_{n+m})$ such that

$$\hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}), \rho_{\max}] & if\ 0 \le \rho_{i,0} \le \sigma, \\ [\sigma, \rho_{\max}] & if\ \sigma \le \rho_{i,0} \le \rho_{\max}, \end{cases} \tag{6}$$

with $i = 1, \ldots, n$, and

$$\hat{\rho}_j \in \begin{cases} [0, \sigma] & if\ 0 \le \rho_{j,0} \le \sigma, \\ \{\rho_{j,0}\} \cup [0, \tau(\rho_{j,0})[& if\ \sigma \le \rho_{j,0} \le \rho_{\max}, \end{cases} \tag{7}$$

with $j = n+1, \ldots, n+m$, where $\tau:[0,1] \to [0,1]$ is the map such that $f(\tau(\rho)) = f(\rho)$ for every $\rho \in [0,1]$ and $\tau(\rho) \ne \rho$ for every $\rho \in [0,1] \setminus \{\sigma\}$.

In order to show the construction of the RS satisfying rules (A) and (B) we recall the following simple cases:
Case 1 - 2×1 junction;
Case 2 - 1×2 junction.
Case 1. We consider the junction in Figure 2 with two incoming roads, 1 and 2, and one outgoing road 3.

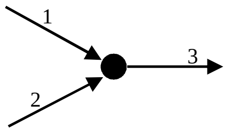

Figure 2: a 2×1 junction.

Fix a right of way parameter $p\in\left]0,1\right[$ describing the percentage of cars crossing the junction. The solution is built as follows. To maximize the through traffic, according to rule (B), we set

$$\hat{\gamma}_3=\min\left\{\hat{\gamma}_1^{\max}+\hat{\gamma}_2^{\max},\hat{\gamma}_3^{\max}\right\}, \tag{8}$$

where $\hat{\gamma}_i^{\max}$, $i=1,2$ and $\hat{\gamma}_3^{\max}$ are respectively given by relations (3) and (4). Observe that A is given by the column vector $(1,1)$. Considering the space (γ_1,γ_2) and the lines

$$\gamma_2=\frac{1-p}{p}\gamma_1, \tag{9}$$

$$\gamma_2+\gamma_1=\hat{\gamma}_3, \tag{10}$$

we indicate with P the intersection point between lines (9) and (10). Therefore the final fluxes must belong to the admissible region

$$\Omega=\left\{(\gamma_1,\gamma_2):0\le\gamma_1+\gamma_2\le\hat{\gamma}_3,0\le\gamma_i\le\gamma_i^{\max},i=1,2\right\}. \tag{11}$$

We distinguish two different cases:
1. P belongs to Ω.
2. P does not belong to Ω.

In the first case (Figure 3) we set $(\hat{\gamma}_1,\hat{\gamma}_2)=P$, while in the second case (Figure 4) we set $(\hat{\gamma}_1,\hat{\gamma}_2)=Q$, where Q is given by the intersection $\Omega\cap\left\{(\gamma_1,\gamma_2):\gamma_1+\gamma_2=\hat{\gamma}_3\right\}$. Once determined $\hat{\gamma}_1,\hat{\gamma}_2,\hat{\gamma}_3$ we are able to compute in a unique way $\hat{\rho}_1,\hat{\rho}_2,\hat{\rho}_3$ applying Theorem 3.

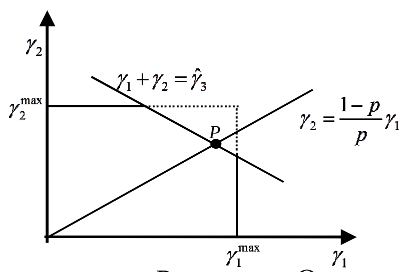

Figure 3: P belongs to Ω.

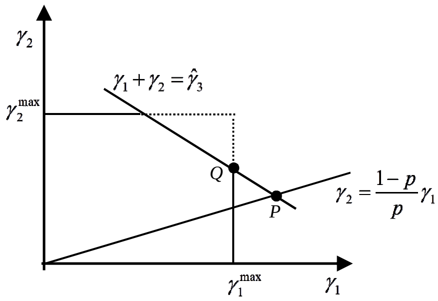

Figure 4: P does not belong to Ω.

Case 2. Deal with the junction in Figure 5 characterized by one incoming road 1 and two outgoing roads, 2 and 3.

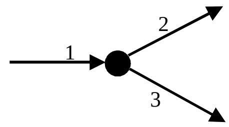

Figure 5: a 1×2 junction.

The rules (A) and (B) are only used. The distribution matrix is given by

$$A=\begin{pmatrix}\alpha\\1-\alpha\end{pmatrix}, \tag{12}$$

where $\alpha\in\left]0,1\right[$ and $(1-\alpha)$ indicate the percentage of cars which, from road 1, goes to roads 2 and 3, respectively. Due to rule (B), the solution to RP is

$$\hat{\gamma}=(\hat{\gamma}_1,\hat{\gamma}_2,\hat{\gamma}_3)=(\hat{\gamma}_1,\alpha\hat{\gamma}_1,(1-\alpha)\hat{\gamma}_1), \tag{13}$$

where $\hat{\gamma}_1=\min\left\{\hat{\gamma}_1^{\max},\frac{\hat{\gamma}_2^{\max}}{\alpha},\frac{\hat{\gamma}_3^{\max}}{1-\alpha}\right\}$. Finally, we determine $\hat{\rho}_1,\hat{\rho}_2,\hat{\rho}_3$ by Theorem 3.

3.1. RS for generalized junctions

Now we focus on some particular cases for a $n\times m$ junction with $n>m$. Starting from the discussion done in subsection 2.1, Case 1, we are able to define a RS, through the archetype of a linear programming (*LP*) problem. Without loss of generality we can analyze the sub-case of a $n\times1$ junction as shown in Figure 6.

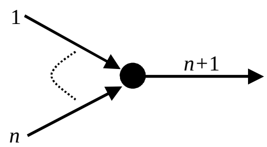

Figure 6: $n\times1$ junction.

For each incoming road we fix a right of way parameter $p_i \in]0,1[$, $i = 1,...,n$, such that $\sum_{i=1}^{n} p_i = 1$. According to rule (B), we set

$$\hat{\gamma}_{n+1} = \max\{\gamma_1 + ... + \gamma_n, \gamma_{n+1}\}, \quad (14)$$

or we can say that the objective function of this problem is $\max(\gamma_1 + ... + \gamma_n)$. The admissible region is given by the set

$$\Omega = \left\{(\gamma_1,..,\gamma_n) : \sum_{i}^{n} \gamma_i \le \hat{\gamma}_{n+1}, 0 \le \gamma_i \le \gamma_i^{\max}, \forall i\right\}. \quad (15)$$

Considering the n-dimensional space $(\gamma_1,...,\gamma_n)$ we can determine the unique solution as the intersection of

$$\gamma_i = \frac{p_i}{p_1}\gamma_1, \forall i = 2,...,n, \quad (16)$$

$$\sum_{i}^{n} \gamma_i \le \hat{\gamma}_{n+1}. \quad (17)$$

We remark that, since no graphical method can be applied, the solution point P is obtained using the simplex method to solve the *LP* problem defined in (15), (16), (17) with objective function in (14).

Then if P belongs to Ω, we set $(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n) = P$, otherwise, $\hat{\gamma}_1 = \gamma_1^{\max}$, and we run the simplex method adding this constraint to the *LP* problem.

Observe that if at some time instant $\gamma_1 = 0$, the solution is given by $(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n) = (0,0,...,0)$. Since this is not acceptable (the through traffic flow is different from zero), in order to determine the correct solution the following micro-algorithm, consisting of three steps, is implemented:

1. Search the incoming road with minimum traffic right of way parameter, except that with zero density.
2. Set the priority constraints, as in (16), referred to the choice at step one.
3. Solve the corresponding *LP* problem.

Let us discuss the case of an $1 \times m$ junction, as in Figure 7, in which the distribution matrix is the following

$$A = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}, \quad (18)$$

with $\alpha_i \in [0,1]$, $i = 1,...,m$, and $\sum_{i=1}^{m} \alpha_i = 1$.

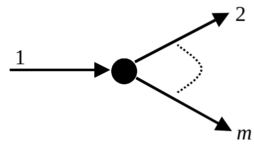

Figure 7: an $1 \times m$ junction.

In this case the flux vector of solution to the *RP* is

$$\hat{\gamma} = (\hat{\gamma}_1, \alpha_2\hat{\gamma}_1, ..., \alpha_m\hat{\gamma}_1), \quad (19)$$

where

$$\hat{\gamma}_1 = \min\left(\gamma_1^{\max}, \frac{\gamma_2^{\max}}{\alpha_2}, ..., \frac{\gamma_m^{\max}}{\alpha_m}\right). \quad (20)$$

Now, we are able to describe the *LP* problem for a $n \times m$ junction assuming $n > m$. In order to satisfy the rules (A), (B) and (C), we get the following *LP* problem:

$$\max \sum_{i=1}^{n} \gamma_i, \quad (21)$$

$$\begin{aligned} &0 \le \sum_{i=1}^{n} \alpha_{1,i}\gamma_i \le \gamma_1^{\max}, \\ &\vdots \\ &0 \le \sum_{i=1}^{n} \alpha_{m,i}\gamma_i \le \gamma_m^{\max}, \end{aligned} \quad (22)$$

$$\gamma_i = \frac{q_i}{q_1}\gamma_1, \ \forall i = 2,...,n, \quad (23)$$

$$0 \le \gamma_i \le \gamma_i^{\max}, \ \forall i = 2,...,n, \quad (24)$$

which can be solved using the simplex method. Then the outgoing fluxes are given by

$$(\hat{\gamma}_{n+1}, ..., \hat{\gamma}_{n+m}) = \left(\sum_{i=1}^{n} \alpha_{1,i}\hat{\gamma}_i, ..., \sum_{i=1}^{n} \alpha_{m,i}\hat{\gamma}_i\right). \quad (25)$$

3.2. RS for time dependent traffic

The real dynamic behavior of drivers at junctions is captured considering time dependent distribution coefficients, which means that, for instance, during a time period of the day, the traffic flows towards some specific direction, while in the successive period towards another one. Then, the matrix A is time dependent. Moreover we include a traffic light on the incoming side of a junction of type 2×2 (Figure 8), where $1, 2$ are the incoming roads and $3, 4$ are the outgoing ones.

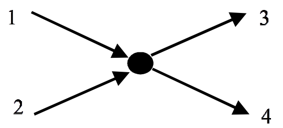

Figure 8: 2×2 junction.

Assume the distribution coefficients are two piecewise constant functions

$$\alpha(t)=\begin{cases}\eta_1 & 0\le t<\tau,\\ \eta_2 & \tau\le t\le T,\end{cases}$$
$$\beta(t)=\begin{cases}\eta_2 & 0\le t<\tau,\\ \eta_1 & \tau\le t\le T,\end{cases} \qquad (26)$$

with $0<\alpha(t)<1$, $0<\beta(t)<1$ and $\alpha(t)\neq\beta(t)$, for each $t\ge 0$. We define two piecewise constant maps as $\chi_1=\chi_1(t)$, $\chi_2=\chi_2(t)$, with $\chi_1(t)+\chi_2(t)=1$ and $\chi_i(t)\in\{0,1\}$, $i=1,2$, for each $t\ge 0$, which represent traffic lights. The values 0 and 1 correspond, respectively, to red and green lights. The matrix A is given by

$$A=\begin{pmatrix}\chi_1(t)\alpha(t) & \chi_2(t)\beta(t)\\ \chi_1(t)(1-\alpha(t)) & \chi_2(t)(1-\beta(t))\end{pmatrix}. \qquad (27)$$

First, let $\chi_i(t)=1, i=1,2$ (no traffic lights) and fix $t<\tau$; to find the solution $(\hat{\gamma}_1,\hat{\gamma}_2)$ at junction we solve the following *LP* problem:

$$\max \gamma_1+\gamma_2,$$
$$\eta_1\gamma_1+\eta_1\gamma_2\le\gamma_3^{\max},$$
$$(1-\eta_1)\gamma_1+(1-\eta_2)\gamma_2\le\gamma_4^{\max}, \qquad (28)$$
$$0\le\gamma_1\le\gamma_1^{\max},$$
$$0\le\gamma_2\le\gamma_2^{\max}.$$

Clearly, for a time-instant belonging to the interval $[\tau,T]$, the *LP* problem will be defined taking account the different values of distribution coefficients, as in (26).

Now, if we suppose that, for some $t<\tau$, $\chi_1(t)=1$ and $\chi_2(t)=0$, i.e. for road 1 the green light is set, while for road 2 red light, the traffic flows from road 1 to roads 3 and 4. In this case the *LP* problem can be reduced to

$$\max \gamma_1,$$
$$\eta_1\gamma_1\le\gamma_3^{\max},$$
$$(1-\eta_1)\gamma_1\le\gamma_4^{\max}, \qquad (29)$$
$$0\le\gamma_1\le\gamma_1^{\max}.$$

4. NUMERICAL METHOD

The space (t,x) is discretized via a numerical grid in $\Re^N\times(0,T)$ using the following notations:

- Δx is the space grid size;
- Δt is the time grid size;
- $(x_m,t_n)=(m\Delta x,n\Delta t)$ for $n\in\mathcal{N}$ and $m\in\mathcal{Z}$ are the grid points.

The values of the velocity v and the density ρ, on the grid are denoted, respectively, by $v_m^n=v(x_m,t_n)$ and $\rho_m^n=\rho(x_m,t_n)$.

In order to find a numerical solution for the conservation law along roads, the Godunov scheme is used. The initial datum ρ_0 is approximated by

$$v_m^0=\frac{1}{\Delta x}\int_{x_m-\frac{1}{2}\Delta x}^{x_m+\frac{1}{2}\Delta x}\rho_0(x)\,dx. \qquad (30)$$

The Godunov scheme is based on exact solutions v^Δ to *RP* at points $\left(m-\frac{1}{2}\right)\Delta x$, $m\in\mathcal{Z}$ and on the projection of the solution

$$v_m^{n+1}=\frac{1}{\Delta x}\int_{x_m-\frac{1}{2}\Delta x}^{x_m+\frac{1}{2}\Delta x}v^\Delta(t_{n+1},x)\,dx. \qquad (31)$$

This procedure can be repeated inductively on every t_n. Under the *CFL* (Courant-Friedrichs-Lewy) condition

$$\Delta t\sup_{m,n}\left\{\sup_{\rho\in I(\rho_m^n,\rho_{m+1}^n)}\left|f'(\rho)\right|\right\}\le\Delta x, \qquad (32)$$

the waves, generated by different *RP*, do not interact. We can use the Gauss-Green formula to compute v^{n+1}. The flux in $x=x_m-\frac{1}{2}\Delta x$ for $t\in(t_n,t_{n+1})$ is given by $f\left(\rho\left(t,x_m-\frac{1}{2}\Delta x\right)\right)=f\left(W_R\left(0;v_{m-1}^n,v_m^n\right)\right)$, where $W_R\left(\frac{x}{t};v_-,v_+\right)$ is the self-similar solution between v_-

and v_+. Similarly for the point $x = x_m + \frac{1}{2}\Delta x$: $f\left(\rho\left(t, x_m + \frac{1}{2}\Delta x\right)\right) = f\left(W_R\left(0; v_m^n, v_{m+1}^n\right)\right)$. As the flux is time invariant and continuous, we can put it out of the integral and, setting $g^G(\rho, v) = f\left(W_R\left(0; \rho, v\right)\right)$ under the condition (35), the scheme can be written as:

$$v_m^{n+1} = v_m^n - \frac{\Delta t}{\Delta x}\left(g^G\left(v_m^n, v_{m+1}^n\right) - g^G\left(v_{m-1}^n, v_m^n\right)\right). \qquad (33)$$

In general the numerical flux of Godunov is

$$g^G(\rho, w) = \begin{cases} \min\limits_{z\in[\rho,w]} f(z), & if\ \rho \le w, \\ \max\limits_{z\in[w,\rho]} f(z), & if\ w \le \rho. \end{cases} \qquad (34)$$

4.1. Boundary condition

Fix a condition at the incoming boundary (incoming flow) $x = 0$: $u(0,t) = \rho_1(t),\ t > 0$, and study equation only for $x > 0$. Inserting a ghost cell, we define the numerical condition as

$$v_0^{n+1} = v_0^n - \frac{\Delta t}{\Delta x}\left(g^G\left(v_0^n, v_1^n\right) - g^G\left(\rho_1^n, v_0^n\right)\right), \qquad (35)$$

where $\rho_1^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \rho_1(t)\,dt$ takes the place of v_{-1}^n. Analogously, the outgoing boundary is defined as follows. Let $x < L = x_N$, then we have

$$v_N^{n+1} = v_N^n - \frac{\Delta t}{\Delta x}\left(g^G\left(v_N^n, \rho_2^n\right) - g^G\left(v_{N-1}^n, v_N^n\right)\right), \qquad (36)$$

where $\rho_2^n = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \rho_2(t)\,dt$ takes the place of v_{N+1}^n that is a ghost cell value.

4.2. Condition at junctions

For roads connected to a junction at the right endpoint we set

$$v_N^{n+1} = v_N^n - \frac{\Delta t}{\Delta x}\left(\hat{\gamma}_i - g^G\left(v_{N-1}^n, v_N^n\right)\right), \qquad (37)$$

while for roads connected to a junction at the left endpoint we have

$$v_0^{n+1} = v_0^n - \frac{\Delta t}{\Delta x}\left(g^G\left(v_0^n, v_1^n\right) - \hat{\gamma}_j\right), \qquad (38)$$

where $\hat{\gamma}_i, \hat{\gamma}_j$ are the flux solutions.

5. SIMULATION RESULTS

In what follows we choose a flux function $f(\rho) = \rho(1-\rho)$ which admits a unique maximum $\sigma = \frac{1}{2}$.

5.1. Simple Junction Scenario

Now we focus on a typical scenario of a 2×2 junction with a traffic light, where $1, 2$ are the incoming roads and $3, 4$ are the outgoing ones, as shown in Figure 8.

Consider the length of each road as normalized, a simulation time interval $[0, T]$ with $T = 10$ (which represents a time horizon of observation) and a numerical grid with $\Delta x = 0.12$ and $\Delta t = CFL \times \Delta x = 0.108$ where $CFL = 0.9$. The number of discrete time instants is given by the ratio $\frac{T}{\Delta t} = 93$ and accordingly the time variable t is referred to these instants. Further we assume the following data:

$$\rho_{1,0} = \rho_{2,0} = \rho_{3,0} = \rho_{4,0} = 0\,,$$
$$\rho_{1,b} = \rho_{2,b} = \rho_{3,b} = \rho_{4,b} = 0.3\,,$$

where for $i = 1, 2, 3, 4$, $\rho_{i,0}$ is the initial density data, $\rho_{i,b}$ is the boundary density data.

From (3) and (4) we get that the maximal fluxes $\gamma_i^{\max}$ are the following:

$$\gamma_1^{\max} = \gamma_2^{\max} = \gamma_3^{\max} = \gamma_4^{\max} = 0.5.$$

The traffic light is modeled by two functions, one for each incoming road, $\chi_1(t)$ and $\chi_2(t)$:

$$\chi_1(t) = \begin{cases} 1 & 0 \le t < 20, \\ 0 & 20 \le t < 40, \\ 1 & 40 \le t < 60, \\ 0 & t \ge 60, \end{cases}$$

$$\chi_2(t) = \begin{cases} 0 & 0 \le t < 20, \\ 1 & 20 \le t < 40, \\ 0 & 40 \le t < 60, \\ 1 & t \ge 60. \end{cases}$$

Finally, for the distribution coefficient α we consider two different cases:

- α constant;
- $\alpha = \alpha(t)$, i.e. time varying.

5.1.1. Case of constant distribution

The distribution matrix is given by

$$A = \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix},$$

which means that the same quantity of cars is distributed on outgoing roads.

The evolution of traffic density on each road is shown in the following figures.

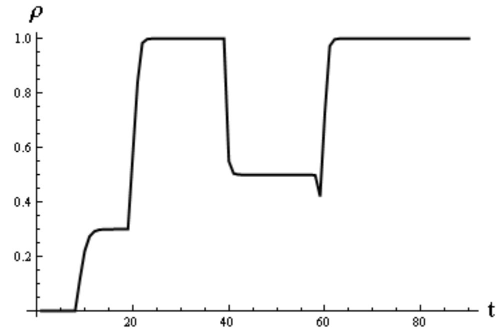

Figure 9: density on road 1.

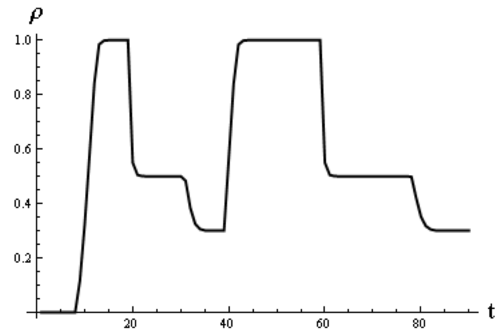

Figure 10: density on road 2.

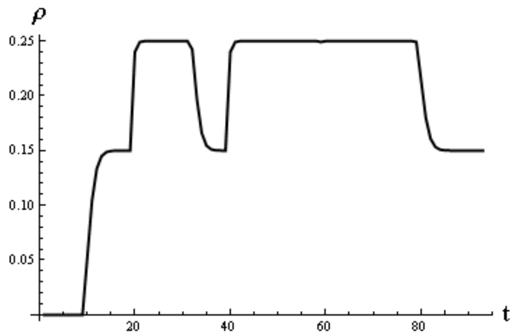

Figure 11: density on roads $3,4$.

As we can see from Figures 9 e 10, in the first time interval $[0,20[$, when the traffic light is green on incoming road 1, ρ_1 increases until the value 0.3, i.e. it reaches the boundary condition, while ρ_2 tends to the value 1, which means that the road 2 is saturated and, consequently, congests. Then, in the successive interval $[20,40[$, when the traffic light is green on road 2, ρ_2 decreases in such way that the road decongests until it reaches the value of density $\sigma = 0.5$, i.e. when the flux attains the maximum. Observe that this behavior is periodic since we choose the same alternate traffic light cycles.

For the outgoing roads (Figure 11), considering the same distribution coefficients, we see that in $[0,20[$ $\rho_3 = \rho_4 = 0.15$ as we expect; in fact from the only incoming road, i.e. road 1, the incoming density is 0.3. Then, in $[20,40[$, since the incoming density from road 2 is equal to 1 but $\gamma_2^{\max} = 0.5$, the corresponding incoming density is $\sigma = 0.5$, so that $\rho_3 = \rho_4 = 0.25$.

5.1.2. Case of time-varying distribution

The distribution matrix is given by

$$A = \begin{pmatrix} \alpha(t) \\ 1-\alpha(t) \end{pmatrix}$$

where $\alpha(t) = -\frac{0.4}{80}t + 0.7.$

Here the evolution of traffic density on the incoming roads is the same of the previous case with constant α, while on the outgoing roads it is shown in the following figures.

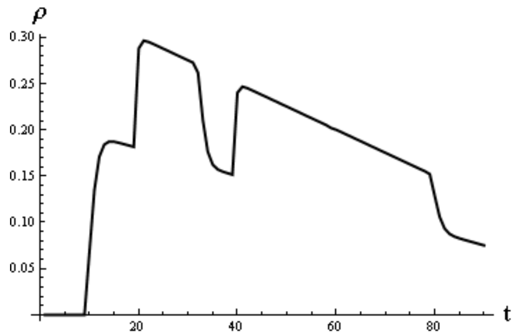

Figure 12: density on road ρ_3.

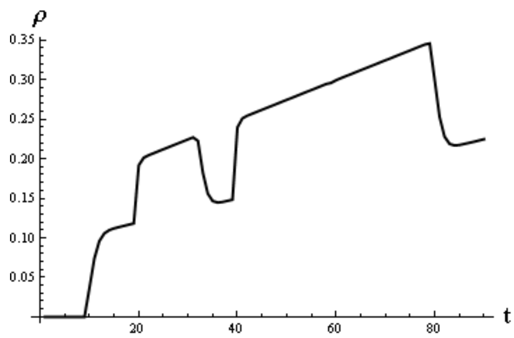

Figure 13: density on road ρ_4.

We can observe that the effect of time-varying distribution consists in a modulation of the whole outgoing traffic. In general, the behavior of the drivers is captured more realistically finding the right dynamic modeling of distribution coefficients.

5.2. Network Scenario

In this section, we study a network scenario with three different junctions:

- $J_1 - 2\times 1$,
- $J_2 - 2\times 2$ with distribution coefficients $\alpha_4 = 0.5$ and $\alpha_6 = 0.5$,
- $J_3 - 3\times 1$ with priority parameters $p_6 = 0.5$, $p_7 = 0.2$ and $p_8 = 0.5$,

linked as in Figure 14.

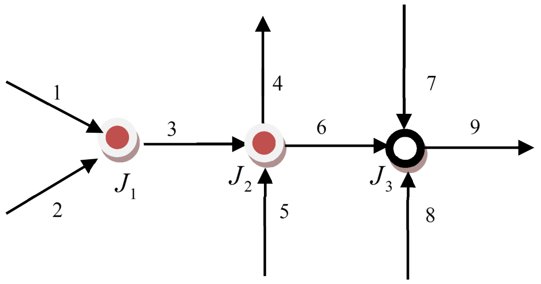

Figure 14: a network with three junctions.

On each road, we set $\rho_{i,0} = 0$, and boundary condition $\rho_b = 0.3$. In accordance to *CFL* condition we set $\Delta x = 0.0123$.

Now, the goal is to compare the effects due to changes of traffic light cycles on the evolution of fluxes outgoing from the network such as to choose the better strategy able to minimize congestion phenomena.

First, we assume in J_1 and J_2 two traffic lights having the green-red cycles as in Table 1.

Table 1: green-red cycles for J_1 and J_2.

	J_1		J_2	
Time instants	**Road 1**	**Road 2**	**Road 3**	**Road 5**
0 - 99	Green	Red	Green	Red
100 - 199	Red	Green	Red	Green
200 - 299	Green	Red	Green	Red
300 - 399	Red	Green	Red	Green
400 - 499	Green	Red	Green	Red
500 - 599	Red	Green	Red	Green
600 - 699	Green	Red	Green	Red
700 - 799	Red	Green	Red	Green
800 - 903	Green	Red	Green	Red

Focusing on the network outgoing roads, i.e. roads 4 and 7, the simulation results of the density evolution are shown, respectively, in Figures 15 and 16.

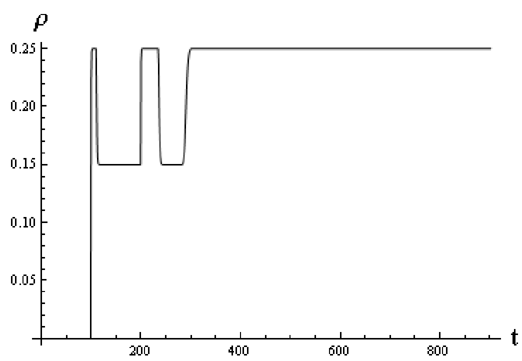

Figure 15: density on road 4.

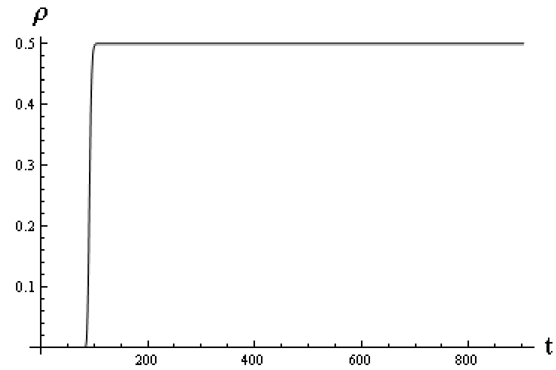

Figure 16: density on road 9.

Now, we change the traffic light cycles as shown in Table 2 and Table 3, setting different time instants.

Table 2: green-red cycles for J_1 .

	J_1	
Time instants	**Road 1**	**Road 2**
0 - 99	Green	Red
100 - 199	Red	Green
200 - 299	Green	Red
300 - 399	Red	Green
400 - 499	Green	Red
500 - 599	Red	Green
600 - 699	Green	Red
700 - 799	Red	Green
800 - 903	Green	Red

Table 3: green-red cycles for J_2.

	J_2	
Time instants	**Road 3**	**Road 5**
0 - 49	Green	Red
50 - 149	Red	Green
150 - 249	Green	Red
250 - 349	Red	Green
350 - 449	Green	Red
450 - 549	Red	Green
550 - 649	Green	Red
650 - 749	Red	Green
750 - 849	Green	Red

In this case the density evolution on roads 4 and 7 is represented in Figures 18 and 19.

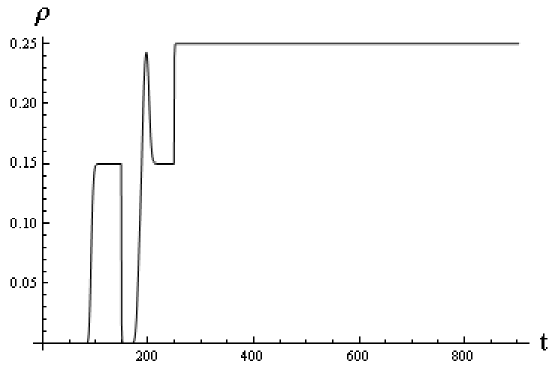

Figure 18: density on road 4.

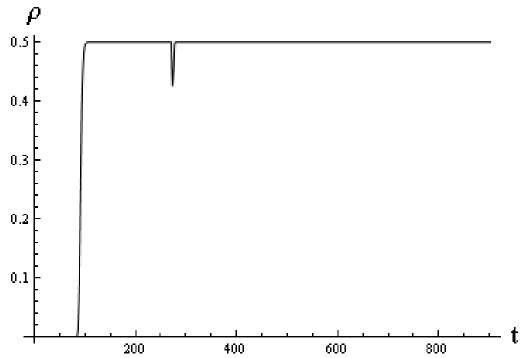

Figure 19: density on road 9.

Notice that, in the second configuration case of traffic lights, the outgoing fluxes are greater than the first case.

Finally, we conclude that, choosing a right policy for the management of fluxes at a junction, playing on distribution coefficients and traffic lights cycles, it is possible to improve traffic conditions and minimize congestion effects.

REFERENCES

Bellomo, N., Coscia, V., 2005. First order models and closure of the mass conservation equation in the mathematical theory of vehicular traffic flow. *C. R. Mec.* 333, pp. 843-851.

Bressan, A., 2000. Hyperbolic Systems of Conservation Laws – The One Dimensional Cauchy Problem. *Oxford Univ. Press*.

Coclite, G., Garavello, M., Piccoli, B., 2005. Traffic flow on road networks. *SIAM Journal on Mathematical Analysis*, vol. 36, no. 6, pp. 1862-1886.

Colombo, R.M, 2002. Hyperbolic phase transitions in traffic flow, *SIAM J. Appl. Math.*, vol. 63, no. 2, pp. 708-721.

Garavello, M., Piccoli, B., 2006. Traffic flow on Networks. *Applied Math. Series*, American Institute of Mathematical Sciences, vol. 1.

Godlewski, E., Raviart, P.A., 1991. Hyperbolic systems of conservation laws, *Mathematics and Applications*, 3-4, Ellipses, Paris.

Godunov, S.K., 1959. A finite difference method for the numerical computation of discontinuous solutions of the equations of fluid dynamics, *Matematicheskii Sbornik*, vol. 47, pp. 271-290.

Helbing, D., 2001. Traffic and related self-driven many particle systems. *Reviews of Modern Physics*, vol. 73, pp. 1067-1141.

Lighthill, M.J., Whitham, G.B., 1955. On kinetic waves. II. Theory of traffic flows on long crowded roads. *Proc. Roy. Soc. London Ser. A*, vol. 229, pp. 317-345.

Richards, P.I., 1956. Shock waves on the highway. *Oper. Res.*, vol. 4, pp. 42-51.

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