STABILITY OF THE CONVEX LINEAR COMBINATION OF FRACTIONAL POSITIVE DISCRETE-TIME LINEAR SYSTEMS

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ABSTRACT
The asymptotic stability of the convex linear combination of fractional positive discrete-time linear systems is addressed. Necessary and sufficient conditions for the asymptotic stability of the convex linear combination are established. The notion of diagonal dominant matrices for nonnegative real matrices is introduced. It is shown that the convex linear combination is asymptotically stable if its matrices are diagonal dominant.

Keywords: Asymptotic stability, convex linear combination, Metzler matrix, nonnegative matrix, positive systems

1. INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs (Farina and Rinaldi 2000, Kaczorek 2002). Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc..

New stability conditions for positive discrete-time linear systems have been proposed by Buslowicz (2008a) and next have been extended to robust stability of fractional discrete-time linear systems in (Buslowicz 2010). The practical stability of positive fractional discrete-time linear systems has been investigated in (Buslowicz and Kaczorek 2009). The stability of positive continuous-time linear systems with delays of the retarded type has been addressed in (Buslowicz 2008b). The independence of the asymptotic stability of positive 2D linear systems with delays of the number and values of the delays has been shown in (Kaczorek 2009b). The asymptotic stability of positive 2D linear systems without and with delays has been considered in (Kaczorek 2009a, 2010a). The stability and stabilization of positive fractional linear systems by state-feedbacks have been analyzed in (Kaczorek 2010b, 2011d). The Hurwitz stability of Metzler matrices has been investigated in (Narendra and Shorten 2010) and some new tests for checking the asymptotic stability of positive 1D and 2D linear systems have been proposed in (Kaczorek 2011b, 201bc). The asymptotic stability of the convex linear combination of positive linear systems has been addressed in (Kaczorek 2012).

In this paper the asymptotic stability of the convex linear combination of fractional positive discrete-time linear systems will be addressed. It will be shown that the convex linear combination is asymptotically stable if its matrices are diagonal dominant.

The paper is organized as follows. In section 2 the basic definition and theorems concerning fractional positive discrete-time linear systems are recalled. The problem is formulated for this class of fractional positive systems in section 3. The problem solution is presented in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.

2. PRELIMINARIES

Consider the fractional discrete-time linear system

$$\Delta^\alpha x_{i+1} = Ax_i, \quad 0 \le \alpha \le 1 \tag{2.1}$$

where $\alpha \in \Re$ is the order, $x_i \in \Re^n$ is the state vector and $A \in \Re^{n\times n}$.

Substituting the fractional difference

$$\Delta^\alpha x_i = \sum_{k=0}^{i} (-1)^k \binom{\alpha}{k} x_{i-k} \tag{2.2a}$$

where

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{for} \quad k=0 \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} & \text{for} \quad k=1,2,\ldots \end{cases} \tag{2.2b}$$

into (2.1), we obtain

$$x_{i+1} = A_\alpha x_i + \sum_{k=2}^{i+1} (-1)^{k+1} \binom{\alpha}{k} x_{i-k+1}\ ,\ A_\alpha = A + I_n \alpha \qquad (2.3)$$

Theorem 2.1. (Kaczorek 2011d) The solution of equation has the form

$$x_i = \Phi_i x_0 \qquad (2.4)$$

where

$$\Phi_{i+1} = \Phi_i A_\alpha + \sum_{k=2}^{i+1} (-1)^{k+1} \binom{\alpha}{k} \Phi_{i-k+1}\ ,\ \Phi_0 = I_n\,. \qquad (2.5)$$

Definition 2.1. The system (2.1) is called (internally) positive fractional system if $x_i \in \Re_+^n$, $i \in Z_+$ for every initial condition $x_0 \in \Re_+^n$.
Theorem 2.2. (Kaczorek 2011d) The fractional system (2.1) is positive if and only if $A_\alpha \in \Re_+^{n\times n}$.
Definition 2.2. The fractional positive system (2.1) is called asymptotically stable if

$$\lim_{i\to\infty} x_i = 0 \ \text{ for all } \ x_0 \in \Re_+^n \qquad (2.6)$$

From (2.4) and (2.6) it follows that the positive fractional system (2.1) is asymptotically stable if and only if

$$\lim_{i\to\infty} \Phi_i = 0 \qquad (2.7)$$

Using (2.5) and (2.7) it is easy to show the following theorem (Kaczorek 2011d).
Theorem 2.3. The fractional positive system (2.1) is asymptotically stable if and only if :

1. The matrix $A + I_n \in \Re_+^{n\times n}$ is asymptotically stable (Schur matrix),
2. The matrix $A_\alpha \in M_n$ is asymptotically stable (Hurwitz Metzler matrix).

Let $\overline{A} = [a_{ij}] \in \Re^{n\times n}$ be a Metzler matrix with negative diagonal entries ($a_{ii} < 0,\ \ i = 1,...,n$) and define

$$\overline{A}_n^{(0)} = \overline{A} = \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1,n}^{(0)} \\ \vdots & \dots & \vdots \\ a_{n,1}^{(0)} & \dots & a_{n,n}^{(0)} \end{bmatrix} = \begin{bmatrix} a_{11}^{(0)} & b_{n-1}^{(0)} \\ c_{n-1}^{(0)} & A_{n-1}^{(0)} \end{bmatrix},$$
$$\overline{A}_{n-1}^{(0)} = \begin{bmatrix} a_{22}^{(0)} & \dots & a_{2,n}^{(0)} \\ \vdots & \dots & \vdots \\ a_{n,2}^{(0)} & \dots & a_{n,n}^{(0)} \end{bmatrix}, \qquad (2.8a)$$
$$b_{n-1}^{(0)} = [a_{12}^{(0)} \ \ \dots \ \ a_{1,n}^{(0)}],\ \ c_{n-1}^{(0)} = \begin{bmatrix} a_{21}^{(0)} \\ \vdots \\ a_{n,1}^{(0)} \end{bmatrix}$$

and

$$\overline{A}_{n-k}^{(k)} = \overline{A}_{n-k}^{(k-1)} - \frac{c_{n-k}^{(k-1)} b_{n-k}^{(k-1)}}{a_{k+1,k+1}^{(k-1)}} = \begin{bmatrix} a_{k+1,k+1}^{(k)} & \dots & a_{k+1,n}^{(k)} \\ \vdots & \dots & \vdots \\ a_{n,k+1}^{(k)} & \dots & a_{n,n}^{(k)} \end{bmatrix}$$
$$= \begin{bmatrix} a_{k+1,k+1}^{(k)} & b_{n-k-1}^{(k)} \\ c_{n-k-1}^{(k)} & A_{n-k-1}^{(k)} \end{bmatrix},$$
$$\overline{A}_{n-k-1}^{(k)} = \begin{bmatrix} a_{k+2,k+2}^{(k)} & \dots & a_{k+2,n}^{(k)} \\ \vdots & \dots & \vdots \\ a_{n,k+2}^{(k)} & \dots & a_{n,n}^{(k)} \end{bmatrix}, \qquad (2.8b)$$
$$b_{n-k-1}^{(k)} = [a_{k+1,k+2}^{(k)} \ \ \dots \ \ a_{k+1,n}^{(k)}],\ \ c_{n-k-1}^{(k)} = \begin{bmatrix} a_{k+2,k+1}^{(k)} \\ \vdots \\ a_{n,k+1}^{(k)} \end{bmatrix}$$

for $k = 1,\dots,n-1$.
It is well-known (Kaczorek 2011c, 2011d) that using the elementary operations we may reduce the matrix

$$\overline{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n} \\ a_{21} & a_{22} & \dots & a_{2,n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix} \qquad (2.9)$$

to the lower triangular form

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & 0 & \dots & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n,1} & \tilde{a}_{n,2} & \dots & \tilde{a}_{n,n} \end{bmatrix}. \qquad (2.10)$$

To check the asymptotic stability of the matrix $\overline{A} = A_\alpha - I_n$ the following theorem is recommended (Kaczorek 2011c, 2011d).
Theorem 2.4. The fractional positive linear system (2.1) for $0 < \alpha < 1$ is asymptotically stable if and only if one of the equivalent conditions is satisfied:

1. All coefficients of the characteristic polynomial

   $$\det[I_n \lambda - \overline{A}] = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 \qquad (2.11)$$

   are positive, i.e. $a_i > 0$ for $i = 1,\dots,n-1$,
2. All principal minors $\Delta_i,\ i = 1,...,n$ of the matrix $-\overline{A} = [-a_{ij}]$ are positive, i.e.

   $$\Delta_1 = -a_{11} > 0,\ \ \Delta_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0,\dots, \qquad (2.12)$$
   $$\Delta_n = \det[-\overline{A}] > 0$$

3. The diagonal entries of the matrices (2.8)

   $$\overline{A}_{n-k}^{(k)} \ \text{ for } k = 1,\dots,n-1 \qquad (2.13)$$

   are negative,

4. The diagonal entries of the lower triangular matrix (2.10) are negative, i.e.

$$\tilde{a}_{kk} < 0 \text{ for } k = 1,\ldots,n \tag{2.14}$$

Proof is given in (Kaczorek 2011c, 2011d).

3. PROBLEM FORMULATION

Consider q positive discrete-time linear systems (2.1) with the matrices

$$A_i = \begin{bmatrix} -a_{11}^{(i)} & a_{12}^{(i)} & \ldots & a_{1,n}^{(i)} \\ a_{21}^{(i)} & -a_{22}^{(i)} & \ldots & a_{2,n}^{(i)} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n,1}^{(i)} & a_{n,2}^{(i)} & \ldots & -a_{n,n}^{(i)} \end{bmatrix} \in \Re_+^{n\times n}, \ i = 1,\ldots,q \tag{3.1}$$

Definition 3.1. The matrix (3.1) is called diagonal dominant if

$$1 - a_{k,k}^{(i)} > \sum_{\substack{j=1 \\ j\neq k}}^{n} a_{k,j}^{(i)} \text{ and } a_{k,j}^{(i)} \geq 0$$

$$\text{for } k, j = 1,\ldots,n;\ i = 1,\ldots,q \tag{3.2}$$

From Definition 3.1 it follows the following lemma.

Lemma 3.1. If the matrices (3.1) are diagonal dominant for $i = 1,\ldots,q$ then the matrix

$$A = \sum_{i=1}^{q} A_i \tag{3.3}$$

is also diagonal dominant.

Definition 3.2. The matrix

$$\hat{A} = \sum_{i=1}^{q} c_i A_i \text{ for } \sum_{i=1}^{q} c_i = 1,\ c_i \geq 0,\ i = 1,\ldots,q \tag{3.4}$$

is called the convex linear combination of the matrices (3.1).

The following question arises: Under which conditions the convex linear combination (3.4) is asymptotically stable nonnegative matrix if the matrices (3.1) are asymptotically stable?

Remark 3.1. The convex linear combination (3.4) of asymptotically stable matrices A_i, $i = 1,\ldots,q$ may be unstable.

For example the convex linear combination

$$\begin{aligned} \hat{A} = cA_1 + (1-c)A_2 &= c\begin{bmatrix} 0.2 & 2 \\ 0 & 0.3 \end{bmatrix} \\ &+ (1-c)\begin{bmatrix} 0.1 & 0 \\ 3 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.1+0.1c & 2c \\ 3-3c & 0.2+0.1c \end{bmatrix} \end{aligned} \tag{3.5}$$

of two asymptotically stable matrices A_1 and A_2 is unstable for $c_1 \leq c \leq c_2$ since for these values of c the characteristic polynomial

$$\begin{aligned} \det[I_n(z+1) - \hat{A}] &= \begin{vmatrix} z+0.9-0.1c & -2c \\ 3c-3 & z+0.8-0.1c \end{vmatrix} \\ &= z^2 + (1.7-0.2c)z + 6.01c^2 - 6.17c + 0.72 \end{aligned} \tag{3.6}$$

has one nonpositive coefficient. Note that

$$c_1 = 0.1342,\ c_2 = 0.8924 \tag{3.7}$$

are the zeros of the polynomial $6.01c^2 - 6.17c + 0.72$.

4. PROBLEM SOLUTION

Theorem 4.1. The convex linear combination (3.4) is asymptotically stable if and only if one of the conditions of Theorem 2.4 is satisfied.

Proof. It is well-known (Kaczorek 2011d, 2012) that the matrix A of positive discrete-time linear systems has eigenvalues in the unit circle if and only if the matrix $A - I_n$ has eigenvalues in the open half of the complex plane. By Theorem 2.4 the convex linear combination (3.4) is asymptotically stable if and only if one of its conditions is satisfied. □

Checking the conditions of Theorem 2.4 for all $c_i \geq 0$ and $\sum_{i=1}^{q} c_i = 1$ is numerically complicated.

Theorem 4.2. The convex linear combination (3.4) is asymptotically stable only if all matrices $A_i \in \Re_+^{n\times n}$ $i = 1,\ldots,q$ of (3.4) are asymptotically stable.

Proof. The asymptotic stability of convex linear combination (3.4) for $c_i = 1$ and $c_1 = \ldots = c_{i-1} = c_{i+1} = \ldots = c_q = 0$ implies the asymptotic stability of the matrix A_i, $i = 1,\ldots,q$. □

Lemma 4.1. Every diagonal dominant nonnegative matrix is asymptotically stable.

Proof. This follows immediately from Gershgorin's theorem since if the condition (3.2) is met for $\hat{A} - I_n$ then all Gershgorin's circles are located in the left half of complex plane. □

Lemma 4.2. If a nonnegative matrix A is asymptotically stable then the matrix cA is also asymptotically stable for $0 < c < 1$.

Proof. Let λ be an eigenvalue of the matrix A and z be an eigenvalue of the matrix cA. Then from the equality

$$\det[I_n z - cA] = \det\left[c\left(I_n \frac{z}{c} - A\right)\right] = c^n \det\left[I_n \frac{z}{c} - A\right] \tag{4.1}$$

we have $z = c\lambda$ and $|z| < 1$ if and only if $|\lambda| < 1$ and $0 < c < 1$. Therefore, the matrix cA is asymptotically stable if and only if the matrix A is asymptotically stable. □

Theorem 4.3. The convex linear combination (3.4) of the upper (or lower) triangular matrices

$$\bar{A}_i = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & \dots & \bar{a}_{1,n} \\ 0 & \bar{a}_{22} & \dots & \bar{a}_{2,n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \bar{a}_{n,n} \end{bmatrix} \in \Re_+^{n\times n},$$

$$\left(\bar{A}_i = \begin{bmatrix} \bar{a}_{11} & 0 & \dots & 0 \\ \bar{a}_{21} & \bar{a}_{22} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ \bar{a}_{n,1} & \bar{a}_{n,2} & \dots & \bar{a}_{n,n} \end{bmatrix} \in \Re_+^{n\times n} \right), \quad i = 1,\dots,q \tag{4.2}$$

is asymptotically stable if and only if the diagonal entries of (4.2) satisfies the condition

$$\bar{a}_{j,j} < 1 \text{ for } j = 1,\dots,n. \tag{4.3}$$

Proof. The matrices (4.2) and $c_i\bar{A}_i$ for $0 < c_i < 1$, $i = 1,\dots,q$ are asymptotically stable if and only if (4.3) holds. Therefore, the convex linear combination (3.4) of upper (lower) triangular matrices (4.2) is asymptotically stable if and only if (4.3) holds. □

From Theorem 4.3 we have the following remark.

Remark 4.2. The convex linear combination (3.4) of the upper (lower) triangular matrices (4.2) is asymptotically stable if and only if the matrices (4.2) are asymptotically stable.

Theorem 4.4. The convex linear combination (3.4) is asymptotically stable if the matrices A_i, $i = 1,\dots,q$ are diagonal dominant.

Proof. Note that if the nonnegative matrix A_i, $i = 1,\dots,q$ is diagonal dominant then the matrix c_iA_i, $c_i > 0$, $i = 1,\dots,q$ is also diagonal dominant for $0 < c_i < 1$; $i = 1,\dots,q$. By Lemma 4.1, and 4.2 the convex linear combination (3.4) is asymptotically stable if the matrices A_i, $i = 1,\dots,q$ are diagonal dominant. □

Example 4.1. Consider the convex linear combination (3.4) for $q = 2$ and the matrices

$$A_1 = \begin{bmatrix} 0.2 & 0.5 \\ 0.1 & 0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.1 & 0.3 \\ 0.4 & 0.2 \end{bmatrix}. \tag{4.4}$$

The matrices (4.4) are diagonal dominant. The convex linear combination (3.4) of (4.4) has the form

$$\hat{A} = cA_1 + (1-c)A_2 = \begin{bmatrix} 0.1+0.1c & 0.3+0.2c \\ 0.4-0.3c & 0.2+0.1c \end{bmatrix} \tag{4.5}$$

and its characteristic polynomial

$$\det[I_2(z+1) - \hat{A}] = \begin{vmatrix} z+0.9-0.1c & -0.3-0.2c \\ -0.4+0.3c & z+0.8-0.1c \end{vmatrix} \tag{4.6}$$
$$= z^2 + (1.7-0.2c)z + 0.07c^2 - 0.16c + 0.6$$

has positive coefficients for $0 < c < 1$.

Therefore, by Theorem 2.4 the convex linear combination (4.5) is asymptotically stable.

Remark 4.3. Note that the convex linear combination (3.4) of the asymptotically stable matrices

$$A_1 = \begin{bmatrix} 0.2 & 2 \\ 0 & 0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.1 & 0 \\ 3 & 0.2 \end{bmatrix}. \tag{4.7}$$

is unstable for $c_1 \le c \le c_2$ (c_1, c_2 are given by (3.7)) since the matrices are not diagonal dominant.

5. CONCLUDING REMARKS

The asymptotic stability of the convex linear combination of asymptotically stable matrices for fractional positive discrete-time linear systems has been addressed. The notion of diagonal dominant matrices for nonegative matrices has been introduced. Necessary and sufficient conditions for the asymptotic stability of the convex linear combinations for fractional positive discrete-time linear systems have been established (Theorem 4.2). Checking the conditions is numerically complicated. It has been shown that the convex linear combinations are asymptotically stable if its matrices are diagonal dominant. The considerations has been illustrated by numerical examples. These considerations can be extended to positive and fractional 2D linear systems.

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