# MANAGEMENT OF SUPPLY NETWORKS USING PDES 

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#### Abstract

The paper presents some numerical results for supply networks modeled by a fluid dynamic approach. A mixed continuum-discrete model is examined: the dynamics on each arch is described by a conservation law for the goods density, and an evolution equation for the processing rate. For solving dynamics at nodes, two routing algorithms are considered, maximizing the flux with possible adjustments of the processing rate. Simulations of a supply network have been made assigning a constant input profile with one discontinuity. A functional is defined to locate the discontinuity point in order to maximize the overall production. Parameters dependence is shown solving Riemann Problems with different rules.


Keywords: conservation laws, supply networks, simulation.

## 1. INTRODUCTION

Control production processes aiming to improve performance in supply networks through the optimal choice of input flow, reduction of dead times and bottlenecks, etc, are of great interest.

Different models have been proposed for supply systems. Most of them are discrete and based on individual parts considerations; others are continuous dealing with ordinary differential equations (see Armbruster et al. 2006a, Armbruster et al. 2006b, Armbruster et al 2004, Daganzo 2003), or/and partial differential equations.

The first paper, that relies on continuous equations, is Armbruster et al. 2006a, where the authors, following a limit procedure on the number of parts and suppliers, have obtained a conservation law, whose flux involves either the goods density or the maximal processing rate.

Due to the difficulties in finding solutions to the general equation proposed in it, other continuous models have been introduced for sequential supply chains (see Bretti et al. 2007, D’Apice et al. 2006, Göttlich et al. 2005), with extensions to networks (Göttlich et al. 2006, D'Apice et al. 2009, Helbing et al. 2004, Helbing et al. 2005).

In this paper, we focus the attention on a discretecontinuous model for supply networks defined in D'Apice et al. 2009. According to it each arch is
modeled by a system of two equations: a conservation law for the goods density, and an evolution equation for the processing rate.

The evolution at a node with one incoming arc and more outgoing ones or with more incoming arcs and one outgoing arc is interpreted thinking of it as a Riemann Problem (RP), a Cauchy Problem with constant initial data on each arc, for the density equation with processing rate data as parameters. RPs are solved using two different "routing" algorithms: the first one allows the redirection of goods to outgoing sub-chains maximizing the flux over incoming sub-chains; the second one is based on the maximization of goods both on incoming and outgoing sub-chains.

Goods flux is maximized for both algorithms also considering two additional rules:

- objects are processed in order to maximize the flux with the minimal value of the processing rate;
- objects are processed in order to maximize the flux: if a solution with only waves in the density exists, then such a solution is taken; otherwise the minimal processing rate wave is produced.
The first rule tends to make adjustments of the processing rate more than the second one, even when it is not necessary for purpose of flux maximization.

Such last rule is more appropriate to reproduce the "Bullwhip effect", see Daganzo 2003: under certain conditions (delays in adaptation of production or delivery rates), the oscillations in delivery and in the resulting inventories (stock level of the products) grow from one producer to the next upstream one, leading to instability with respect to perturbation in the production rate.

The model can be used to study situations characterized by the possibility to reorganize the supply system: in particular, the processing rate can be readapted for some contingent necessity.

Using some ad hoc numerical schemes (see Bretti et al. 2007), based on the classical Godunov method (Godunov, 1959), simulation results have been obtained for a supply network modeling the chips production. In particular, a piecewise constant function with one discontinuity, namely a function of Heavyside type, has been chosen as input profile. In fact, as it happens in
real processes, goods are injected inside supply networks at almost constant levels in different time intervals. The obtained results present some expected features and some unexpected ones: the production, measured by the density on the last arc of the chain, is strongly influenced by the discontinuity point of the input profile; unexpectedly, analysis on the final product flow indicates that final goods start to be produced always at the same temporal instant, independently from the choice of the discontinuity in the input profile. Moreover, discontinuity shifts do not imply simply temporal translations of final product flows, hence indicating the presence of a strong non linearity for the whole system. Finally, a numerical study of the temporal integral of the final product flow (representing the number of produced goods) shows the existence of a time instant at which the discontinuity point of the input profile has to be placed for the maximization of the overall production. This value does not depend on the rules used to solve the dynamics at nodes.

The outline of the paper is the following. Section 2 deals with the mathematical model for supply networks. Riemann Solvers at nodes are described considering different routing algorithms, and finally an example is reported. In Section 3, the numerical results obtained for a supply network are considered and discussed. Finally, the paper ends with conclusions in Section 4.

## 2. MATHEMATICAL MODEL

A supply network is a finite connected graph consisting of a finite set of arcs (sub-chains) $I=\left\{I_{k}: k=1, \ldots, N+1\right\}$ and a finite set of junctions $P$.

On each sub-chain $I_{k}$ (see D'Apice et al. 2009) we consider the system:
$\left\{\begin{array}{l}\left(\rho_{k}\right)_{t}+f_{\varepsilon}^{k}\left(\rho_{k}, \mu_{k}\right)_{x}=0, \\ \left(\mu_{k}\right)_{t}-\left(\mu_{k}\right)_{x}=0,\end{array}\right.$
where $\rho_{k}(t, x)$ and $\mu_{k}(t, x)$ are, respectively, the density of the processed objects on $I_{k}$ and the production rate of $I_{k}$, while $f_{\varepsilon}^{k}$ is the flux, defined as follows:

$$
f_{\varepsilon}^{k}\left(\rho_{k}, \mu_{k}\right)= \begin{cases}\rho_{k}, & 0 \leq \rho_{k} \leq \mu_{k},  \tag{2}\\ \mu_{k}+\varepsilon\left(\rho_{k}-\mu_{k}\right), & \mu_{k} \leq \rho_{k} \leq \rho_{k}^{\max }\end{cases}
$$

or, alternatively,
$f_{\varepsilon}^{k}\left(\rho_{k}, \mu_{k}\right)= \begin{cases}\varepsilon \rho_{k}+(1-\varepsilon) \mu_{k}, & 0 \leq \mu_{k} \leq \rho_{k}, \\ \rho_{k}, & \rho_{k} \leq \mu_{k} \leq \mu_{k}^{\max },\end{cases}$
where $\rho_{k}^{\max }$ and $\mu_{k}^{\max }$ are, respectively, the maximum density and processing rate. From now on, we assume
that $\varepsilon$ is fixed and, for simplicity, we drop the indices, thus indicating the flux by $f\left(\rho_{k}, \mu_{k}\right)$.

Remark We can consider different fluxes $f_{\varepsilon_{k}}^{k}$ for each sub-chain $I_{k}$ (also choosing $\varepsilon$ dependent on $k$ ), or different slopes $m_{k}$ for each sub-chain $I_{k}$ :
$f_{\varepsilon}^{k}\left(\rho_{k}, \mu_{k}\right)= \begin{cases}m_{k} \rho_{k}, & 0 \leq \rho_{k} \leq \frac{\mu_{k}}{m_{k}}, \\ \mu_{k}+\varepsilon\left(m_{k} \rho_{k}-\mu_{k}\right), & \frac{\mu_{k}}{m_{k}} \leq \rho_{k} \leq \rho_{k}^{\max },\end{cases}$
where $m_{k} \geq 0$ represents the velocity of each processor and is given by:
$m_{k}=\frac{L_{k}}{T_{k}}$,
with $L_{k}$ and $T_{k}$, respectively, fixed length and processing time of processor $I_{k}$.

Sub-chains are connected by junctions $P$, each one having a finite number of incoming sub-chains and outgoing ones. Hence, we identify $P$ with $\left(\left(i_{1}, \ldots, i_{n}\right),\left(j_{1}, \ldots j_{m}\right)\right)$ where the first $n$-tuple indicates the set of incoming sub-chains and the second $m$-tuple the set of outgoing sub-chains. Each sub-chain can be incoming sub-chain at most for one junction and outgoing at most for one junction.


Figure 1: Junction $P$ with $n$ incoming sub-chains and $m$ outgoing ones

The supply network evolution is described on each arc $I_{k}$ by a finite set of functions $\left(\rho_{k}, \mu_{k}\right)$ defined on $\left[0,+\infty\left[\times I_{k}\right.\right.$. Dynamics at a junction is obtained solving RPs.

Definition A Riemann Solver (RS) for the junction $P$ with $n$ incoming sub-chains and $m$ outgoing ones (of $n \times m$ type) is a map that associates to a Riemann data $\left(\rho_{0}, \mu_{0}\right)=\left(\rho_{1,0}, \mu_{1,0}, \ldots, \rho_{n+m, 0}, \mu_{n+m, 0}\right)$ at $P$ a vector $\left(\hat{\rho}_{0}, \hat{\mu}_{0}\right)=\left(\hat{\rho}_{1}, \hat{\mu}_{1}, \ldots, \hat{\rho}_{n+m}, \hat{\mu}_{n+m}\right)$ so that the solution is given by the waves $\left(\rho_{i, 0}, \hat{\rho}_{i}\right)$ and $\left(\mu_{i, 0}, \hat{\mu}_{i}\right)$ on the sub-
chain $I_{i}, i=1, \ldots, n$ and by the waves $\left(\hat{\rho}_{j}, \rho_{j, 0}\right)$ on the sub-chain $I_{j}, j=n+1, \ldots, n+m$. We require the consistency condition $R S\left(R S\left(\left(\rho_{0}, \mu_{0}\right)\right)\right)=R S\left(\left(\rho_{0}, \mu_{0}\right)\right)$.

### 2.1. Riemann Solvers for suppliers

We discuss RSs for two types of nodes, according to the real case we examine here (for more detail refer to Bretti et al. 2007 and D'Apice et al. 2009):

1. a node with two incoming sub-chains and one outgoing one ( $2 \times 1$ );
2. a node with one incoming sub-chain and two outgoing ones $(1 \times 2)$.

For a given arc $I_{k},(1)$ is a system of conservation laws in the variables $U=(\rho, \mu)$, namely:
$U_{t}+F(U)_{x}=0$,
with flux function

$$
\begin{equation*}
F(U)=(f(\rho, \mu),-\mu) . \tag{7}
\end{equation*}
$$

Eigenvalues and eigenvectors are:
$\lambda_{1}(\rho, \mu) \equiv-1, \quad r_{1}(\rho, \mu)=\left\{\begin{array}{cl}\binom{0}{1}, & \rho<\mu, \\ \binom{-\frac{1-\varepsilon}{1+\varepsilon}}{1}, & \rho>\mu,\end{array}\right.$
$\lambda_{2}(\rho, \mu)=\left\{\begin{array}{ll}1, & \rho<\mu, \\ \varepsilon, & \rho>\mu,\end{array} \quad r_{2}(\rho, \mu)=\binom{1}{0}\right.$.
Hence the Hugoniot curves for the first family are vertical lines above the secant $\rho=\mu$ and lines with slope close to $-1 / 2$ below the same secant. The Hugoniot curves for the second family are just horizontal lines. Since we consider positive and bounded values for the variables, we fix the invariant region:

$$
\begin{align*}
& D=\left\{(\rho, \mu): 0 \leq \rho \leq \rho_{\max }, 0 \leq \mu \leq \mu_{\max }\right. \\
& \left.0 \leq(1+\varepsilon) \rho+(1-\varepsilon) \mu \leq(1+\varepsilon) \rho_{\max }=2 \mu_{\max }\right\} . \tag{10}
\end{align*}
$$

Observe that:

$$
\begin{equation*}
\rho_{\max }=\mu_{\max } \frac{2}{1+\varepsilon} . \tag{11}
\end{equation*}
$$

We consider a node $P$ of $n \times m$ type and a Riemann initial datum $\left(\rho_{1,0}, \mu_{1,0}, \ldots, \rho_{n+m, 0}, \mu_{n+m, 0}\right)$. The following Lemma holds:

Lemma On the incoming sub-chains, only waves of the first family may be produced, while on the outgoing sub-chains only waves of the second family may be produced.

From such Lemma, given the initial datum, for every RS it follows that:

$$
\begin{array}{ll}
\hat{\rho}_{i}=\varphi\left(\hat{\mu}_{i}\right), & i=1, \ldots, n  \tag{12}\\
\hat{\mu}_{j}=\mu_{j, 0}, & j=n+1, \ldots, n+m
\end{array}
$$

where the function $\varphi(\cdot)$ describes the first family curve through $\left(\rho_{k, 0}, \mu_{k, 0}\right)$ as function of $\hat{\mu}_{k}$ :
$\varphi\left(\hat{\mu}_{k}\right)= \begin{cases}\bar{\mu}_{k}, & \hat{\mu}_{k} \geq \bar{\mu}_{k}, \\ \frac{(\varepsilon-1) \hat{\mu}_{k}+2 \rho_{k, 0}}{1+\varepsilon}, & \hat{\mu}_{k}<\bar{\mu}_{k}, \rho_{k, 0} \leq \mu_{k, 0}, \\ \frac{(\varepsilon-1)\left(\hat{\mu}_{k}-\mu_{k, 0}\right)}{1+\varepsilon}+\rho_{k, 0}, & \hat{\mu}_{k}<\bar{\mu}_{k}, \rho_{k, 0}>\mu_{k, 0},\end{cases}$
where $\bar{\mu}_{k}$ is the point at which the first family curve changes:
$\bar{\mu}_{k}= \begin{cases}\rho_{k, 0}, & \rho_{k, 0} \leq \mu_{k, 0}, \\ \frac{1+\varepsilon}{2} \rho_{k, 0}+\frac{1-\varepsilon}{2} \mu_{k, 0}, & \rho_{k, 0}>\mu_{k, 0} .\end{cases}$

We define two different RSs at a junction to represent two different routing algorithms:
RA1. We assume that:
(A) the flow from incoming sub-chains is distributed on outgoing ones according to fixed coefficients;
(B) respecting (A), the processor chooses to process goods in order to maximize fluxes (i.e., the number of goods which are processed) on incoming sub-chains.

RA2. We assume that the number of goods through the junction is maximized both over incoming and outgoing sub-chains.

For both routing algorithms we can maximize the flux of goods considering one of the two additional rules:
SC2. The objects are processed in order to maximize the flux with the minimal value of the processing rate.
SC3. The objects are processed in order to maximize the flux. If a solution with only waves in the density $\rho$ exists, then such solution is taken, otherwise the minimal $\mu$ wave is produced.

To define RPs according to rules RA1 and RA2, we introduce the notation:
$f_{k}=f\left(\rho_{k}, \mu_{k}\right)$,
and define the maximum flux that can be obtained by a wave solution on each production sub-chain:
$f_{k}^{\max }= \begin{cases}\bar{\mu}_{k}, & k=1, \ldots, n, \\ \mu_{k, 0}+\varepsilon\left(\rho_{\max }-\mu_{k, 0} \frac{\rho_{\max }-\mu_{\max }}{\mu_{\max }}-\mu_{k, 0}\right), & k=n+1, \ldots, n+m .\end{cases}$

It is possible to prove that a necessary and sufficient condition for the solvability of RPs at nodes is

$$
\begin{equation*}
\sum_{i=1}^{n} f_{i}^{\min } \leq \sum_{j=n+1}^{n+m}\left[\mu_{j, 0}+\varepsilon\left(\rho_{\max }-\mu_{j, 0} \frac{\rho_{\max }-\mu_{\max }}{\mu_{\max }}-\mu_{j, 0}\right)\right] \tag{17}
\end{equation*}
$$

where
$f_{i}^{\min }\left(\left(\rho_{0}, \mu_{0}\right)\right)= \begin{cases}\frac{2 \varepsilon}{1+\varepsilon} \rho_{0}, & \rho_{0} \leq \mu_{0}, \\ \varepsilon \rho_{0}+\frac{\varepsilon(1-\varepsilon)}{1+\varepsilon} \mu_{0}, & \rho_{0}>\mu_{0} .\end{cases}$

### 2.1.1. One outgoing sub-chain

In this case, algorithms RA1 and RA2 coincide since there is only one outgoing sub-chain.

We fix a node $P$ with 2 incoming arcs (labelled by 1 and 2 ) and 1 outgoing one (indicated by 3 ) and a Riemann initial datum given by $\left(\rho_{0}, \mu_{0}\right)=\left(\rho_{1,0}, \mu_{1,0}, \rho_{2,0}, \mu_{2,0}, \rho_{3,0}, \mu_{3,0}\right)$. Let us denote with $(\hat{\rho}, \hat{\mu})=\left(\hat{\rho}_{1}, \hat{\mu}_{1}, \hat{\rho}_{2}, \hat{\mu}_{2}, \hat{\rho}_{3}, \hat{\mu}_{3}\right)$ the solution of the RP at $P$. We introduce a priority parameter $q \in] 0,1[$, that indicates a level of priority at the junction of incoming sub-chains. We define:
$\Gamma=\min \left\{\Gamma_{\text {inc }}, \Gamma_{\text {out }}\right\}$,
where
$\Gamma_{\text {inc }}=\sum_{i=1}^{2} f_{i}^{\max }, \Gamma_{\text {out }}=f_{3}^{\max }$.

First, we compute $\hat{f_{i}}, i=1,2,3$ according to rules (SC2) and (SC3). Introduce the conditions:
(A1) $q f_{3}^{\max }<f_{1}^{\max }$;
(A2) $(1-q) f_{3}^{\max }<f_{2}^{\max }$.
If $\Gamma=\Gamma_{i n c}$, we get that $\hat{f}_{i}=f_{i}^{\max }, \quad i=1,2$,
$\hat{f}_{3}=f_{1}^{\max }+f_{2}^{\max }$.
If $\Gamma<\Gamma_{i n c}$, we have that:

- $\hat{f}_{1}=q f_{3}^{\max }, \hat{f}_{2}=(1-q) f_{3}^{\max }, \hat{f}_{3}=f_{3}^{\max }$ when A1 and A2 are both satisfied;
- $\hat{f}_{1}=f_{3}^{\max }-f_{2}^{\max }, \hat{f}_{2}=f_{2}^{\max }, \hat{f}_{3}=f_{3}^{\max }$ when A1 holds and A2 is not satisfied;
- $\hat{f}_{1}=f_{1}^{\text {max }}, \hat{f}_{2}=f_{3}^{\text {max }}-f_{1}^{\max }, \hat{f}_{3}=f_{3}^{\max }$ when A1 is not satisfied and A2 holds.
The case of both A1 and A2 false is not possible, since it would be $f_{3}^{\max }>\Gamma_{i n c}$.

Now, we compute $\hat{\rho}_{k}$ and $\hat{\mu}_{k}, k=1,2,3$. On the incoming sub-chains $i, i=1,2$, we have to distinguish two subcases.

If $\hat{f}_{i}=f_{i}^{\text {max }}$, according to rules SC 2 and SC 3 , we get:

If $\hat{f}_{i}<f_{i}^{\text {max }}$, for both SC 2 and SC 3 rules, we get that $\hat{\mu}_{i}, i=1,2$, solves the equation:
$\hat{\mu}_{i}+\varepsilon\left(\varphi\left(\hat{\mu}_{i}\right)-\hat{\mu}_{i}\right)=\hat{f}_{i}$,
while

$$
\begin{equation*}
\hat{\rho}_{i}=\varphi\left(\hat{\mu}_{i}\right), i=1,2 . \tag{23}
\end{equation*}
$$

On the outgoing sub-chain we have, for both rules SC2 and SC3:
$\hat{\mu}_{3}=\mu_{3,0}$,
while $\hat{\rho}_{3}$ is the unique value solving the equation $f_{3}\left(\mu_{3,0}, \hat{\rho}_{3}\right)=\hat{f}_{3}$, namely:
$\hat{\rho}_{3}= \begin{cases}\hat{f}_{3}, & \hat{f}_{3} \leq \mu_{3,0}, \\ \frac{\hat{f}_{3}-\mu_{3,0}}{\varepsilon}+\mu_{3,0}, & \hat{f}_{3}>\mu_{3,0} .\end{cases}$

### 2.1.2. One incoming sub-chain

Consider a node $P$ with 1 incoming arc, labelled by 1 , and 2 outgoing ones, indicated by 2 and 3 and an initial datum $\left(\rho_{0}, \mu_{0}\right)=\left(\rho_{1,0}, \mu_{1,0}, \rho_{2,0}, \mu_{2,0}, \rho_{3,0}, \mu_{3,0}\right)$.

We introduce a distribution parameter $\alpha \in] 0,1[$, that indicates the percentage of goods, which, from the incoming arc 1 , is directed to the outgoing arc 2 (obviously, the arc 3 is interested by a percentage of goods equal to $1-\alpha$ ). We have different solutions for algorithms RA1 and RA2. In what follows, the asymptotic solution is reported only for the RA1 algorithm, since RA2 is solved as for the node with one outgoing sub-chain.

As usual, we first compute the fluxes solutions. Following rules (A) and (B) of the algorithm RA1, we get that:
$\hat{f}_{1}=\min \left\{f_{1}^{\max }, \frac{f_{2}^{\max }}{\alpha}, \frac{f_{3}^{\max }}{1-\alpha}\right\}, \hat{f}_{2}=\alpha \hat{f}_{1}, \hat{f}_{3}=(1-\alpha) \hat{f}_{1}$.

Densities and processing rates, $\hat{\rho}_{i}$, and $\hat{\mu}_{i}, i=1,2,3$, are obtained as follows.
If $\hat{f}_{1}=f_{1}^{\text {max }}$, we get:
$S C 2: \begin{aligned} & \hat{\rho}_{1}=\bar{\mu}_{1}, \\ & \hat{\mu}_{1}=\bar{\mu}_{1},\end{aligned} \quad S C 3: \begin{aligned} & \hat{\rho}_{1}=\bar{\mu}_{1}, \\ & \hat{\mu}_{1}=\max \left\{\bar{\mu}_{1}, \mu_{1,0}\right\} .\end{aligned}$

If $\hat{f}_{1}<f_{1}^{\text {max }}, \hat{\mu}_{1}$, either for rule SC 2 or SC 3 , satisfies the equation:
$\hat{\mu}_{1}+\varepsilon\left(\varphi\left(\hat{\mu}_{1}\right)-\hat{\mu}_{1}\right)=\hat{f}_{1}$,
while:
$\hat{\rho}_{1}=\varphi\left(\hat{\mu}_{1}\right)$.
On the outgoing sub-chain $j, j=2,3$, for both rules SC 2 and SC3, we have that:
$\hat{\mu}_{j}=\mu_{j, 0}$,
while $\hat{\rho}_{j}$ solves the equation $f_{j}\left(\mu_{j, 0}, \hat{\rho}_{j}\right)=\hat{f}_{j}$, namely:
$\hat{\rho}_{j}= \begin{cases}\hat{f}_{j}, & \hat{f}_{j} \leq \mu_{j, 0}, \\ \frac{\hat{f}_{j}-\mu_{j, 0}}{\varepsilon}+\mu_{j, 0}, & \hat{f}_{j}>\mu_{j, 0} .\end{cases}$
Remark For sequential sub-chains (one incoming arc, 1, and one outgoing arc, 2), the fluxes solutions are $\hat{f}_{1}=\hat{f}_{2}=\min \left\{f_{1}^{\max }, f_{2}^{\max }\right\}$ while $\hat{\rho}_{i}$, and $\hat{\mu}_{i}, i=1,2$, are obtained for rules SC2 and SC3 as before.

### 2.2. Example

In what follows we report densities and production rates at the instant $t=0$ and after some times (at $t=1$ ) for different initial data using different routing algorithms.

We consider a node of type $2 \times 1$, assuming the following data:
$\varepsilon=0.25, \mu_{i}^{\max }=0.8, i=1,2,3$,
$\left(\rho_{1,0}, \rho_{2,0}, \rho_{3,0}\right)=(0.35,0.2,0.6)$,
$\left(\mu_{1,0}, \mu_{2,0}, \mu_{3,0}\right)=(0.95,0.55,0.3)$.

As there is only one outgoing sub-chain, algorithms RA1 and RA2 coincide and the choice
$q=0.6$ indicates that $60 \%$ of goods flow is directed from arc 1 to the outgoing one. In Table 1, numerical results for asymptotic fluxes, densities and production rates are reported while Figures 1 and 2 show the behaviour of density and production rate waves. For both rules SC 2 and SC 3 , the results are the same, with the exception of values $\hat{\mu}_{1}$ and $\hat{\mu}_{2}$ for rule SC3. For sub-chain 3 , a shock wave in the density connect the initial and the asymptotic state while, for sub-chain 1 and 2, there is no waves formation (Figure 2). A similar situation happens for production rates (Figure 3): in the case SC2, only sub-chains 1 and 2 are interested by waves formation. For rule SC3, shock formations do not occurs, as all sub-chains have asymptotic states equal to the initial ones. In fact SC2 tends to make adjustments of the processing rate more than SC3.

Table 1: Numerical results for a node of $2 \times 1$ type

| RA1 = RA2 |  |  |
| :---: | :---: | :---: |
|  | SC2 | SC3 |
| $\hat{f}$ | $(0.35,0.2,0.55)$ | $(0.35,0.2,0.55)$ |
| $\hat{\rho}$ | $(0.35,0.2,1.01)$ | $(0.35,0.2,1.01)$ |
| $\hat{\mu}$ | $(0.35,0.2,0.3)$ | $(0.95,0.55,0.3)$ |



Figure 2: Densities at $t=0$ and $t=1$ on sub-chains for rules SC 2 and SC 3


Figure 3: Production rates at $t=0$ and $t=1$ on subchains for rule SC 2

## 3. SIMULATIONS

In this section, we present some simulation results to foresee the behaviour of goods fluxes on a supply network. In particular, we study how to choose the injection times of different goods levels to increase the production.

### 3.1. Numerical methods

We refer to a Godunov method for a $2 \times 2$ system (details are in Bretti et al. 2007, Godunov 1959), which
is described as follows. Define a discrete grid in the plane $(x, t)$, whose points are $\left(x_{j}, t^{n}\right)=(j \Delta x, n \Delta t)$, $j \in \square, \quad n \in \square$, and indicate by ${ }^{k} \rho_{j}^{n}$ and ${ }^{k} \mu_{j}^{n}$, respectively, the approximations of density and production rate of the arc $I_{k}$ in the point $\left(x_{j}, t^{n}\right)$. An approximation scheme for the system (1) reads as:
$\left\{\begin{array}{l}k \rho_{j}^{n+1}={ }^{k} \rho_{j}^{n}-\frac{\Delta t}{\Delta x}\left(g\left({ }^{k} \rho_{j}^{n},{ }^{k} \rho_{j+1}^{n}\right)-g\left({ }^{k} \rho_{j-1}^{n},{ }^{k} \rho_{j}^{n}\right)\right), \\ { }^{k} \mu_{j}^{n+1}={ }^{k} \mu_{j}^{n}+\frac{\Delta t}{\Delta x}\left({ }^{k} \mu_{j+1}^{n}-{ }^{k} \mu_{j}^{n}\right),\end{array}\right.$
where the Godunov numerical flux $g$ is found solving RPs among the states $\left(\rho_{-}, \mu_{-}\right)$on the left and $\left(\rho_{+}, \mu_{+}\right)$on the right:

$$
\begin{align*}
& g\left(\rho_{-}, \mu_{-}, \rho_{+}, \mu_{+}\right)= \\
& = \begin{cases}\left(\rho_{-},-\mu_{+}\right), & \rho_{-}<\mu_{-}, \rho_{-} \leq \mu_{+}, \\
\left(\frac{1-\varepsilon}{1+\varepsilon} \mu_{+}+\frac{2 \varepsilon}{1+\varepsilon} \rho_{-},-\mu_{+}\right), & \rho_{-}<\mu_{-}, \rho_{-}>\mu_{+}, \\
\left(\frac{1+\varepsilon}{2} \rho_{-}+\frac{1-\varepsilon}{2} \mu_{-},-\mu_{+}\right), & \rho_{-} \geq \mu_{-}, \mu_{+}>\tilde{\mu}, \\
\left(\frac{1-\varepsilon}{1+\varepsilon}\left(\mu_{+}+\varepsilon \mu_{-}\right)+\varepsilon \rho_{-},-\mu_{+}\right), & \rho_{-} \geq \mu_{-}, \mu_{+} \leq \tilde{\mu},\end{cases} \tag{34}
\end{align*}
$$

with
$\tilde{\mu}=\mu_{-}+\frac{1+\varepsilon}{2}\left(\rho_{-}-\mu_{-}\right)$.
We need to introduce the boundary data value, given by the term ${ }^{k} \rho_{j-1}^{n}$. For the first arc of the supply network, ${ }^{k} \rho_{j-1}^{n}$ is defined by an assigned input profile; otherwise, ${ }^{k} \rho_{j-1}^{n}$ is determined by the solution to RPs at nodes.

Remark. The construction of the Godunov method is based on the exact solution to the RP in the cell $] x_{j-1}, x_{j}[\times] t^{n}, t^{n+1}[$. To avoid the interaction of waves in two neighbouring cells before time $\Delta t$, we impose a CFL condition like:

$$
\begin{equation*}
\frac{\Delta t}{\Delta x} \max \left\{\left|\lambda_{0}, \lambda_{1}\right|\right\} \leq \frac{1}{2}, \tag{36}
\end{equation*}
$$

where $\lambda_{0}$ and $\lambda_{1}$ are the eigenvalues of system (1). Since, in this case, the eigenvalues are such that $\left|\lambda_{0}\right|=1,\left|\lambda_{1}\right| \leq 1$, the CFL condition reads as:
$\frac{\Delta t}{\Delta x} \leq \frac{1}{2}$.

### 3.2. A complex network

We present some simulation results for a supply network, whose topology is in Figure 4.


Figure 4: Network with 8 nodes and 10 arcs
Such a network can model the chips production. First, potatoes are washed (arc $I_{1}$ ) and then they are skinned off (arc $I_{2}$ ). Assuming that two different types of fried potatoes are produced (classical and stick, for example), node 2 is a diverging point: a percentage $\alpha$ of potatoes are sent to arc $I_{3}$ for stick chips production, and a percentage $1-\alpha$ to arc $I_{4}$ for the classical potatoes production. On arcs $I_{5}$ and $I_{6}$, potatoes are fried and on arcs $I_{7}$ and $I_{8}$ they are salted. Node 7 is a merging point: considering a certain priority level $q$, potatoes are directed to arc $I_{9}$ where they are put in envelope; on arc $I_{10}$, the obtained packets are sealed.

The goods evolution inside the supply network is simulated in a time interval $[0, T]$, with $T=1000 \mathrm{~min}$, using the approximation scheme (33) with $\frac{\Delta t}{\Delta x}=\frac{1}{2}$. The dynamics at node 2 is solved using the RA1 algorithm. In fact, the redirection of potatoes in order to maximize the production on both incoming and outgoing sub-chains is not possible, since classical and stick potatoes have different shapes. Moreover, at node 2, we use rule SC 2 and a distribution coefficient $\alpha=0.3$ for arc $I_{3}$. At node 7, dynamics is solved using the RA1 algorithm with rule SC3 and priority level $q=0.4$ for arc $I_{7}$ (notice that, for such last node, algorithm RA1 and RA2 coincide).

We assume that, at the beginning of the simulation $(t=0)$, all arcs are empty. Moreover, in Table 2, initial conditions for processing rates, maximal processing rates, lengths and processing times, are reported for each arc $I_{k}, k=1, \ldots, 10$.

Table 2: Parameters for the supply network

| $I_{k}$ | $\mu_{k, 0}$ | $\mu_{k}^{\max }$ | $L_{k}$ | $T_{k}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 10 | 15 | 15 | 15 |
| 2 | 7 | 10 | 30 | 30 |
| 3 | 7 | 10 | 20 | 20 |


| 4 | 15 | 20 | 15 | 15 |
| :---: | :---: | :---: | :---: | :---: |
| 5 | 5 | 8 | 20 | 20 |
| 6 | 5 | 10 | 20 | 20 |
| 7 | 12 | 12 | 20 | 20 |
| 8 | 10 | 10 | 25 | 25 |
| 9 | 15 | 15 | 15 | 15 |
| 10 | 10 | 10 | 10 | 10 |

Maximal densities on arcs are obtained using equation (11), where we consider $\varepsilon=0.2$. Boundary data are also needed: for arc 1 , it represents the amount of goods, that have to be processed inside the supply network; for arc 8 , it is a sort of wished production.

The input profile for arc 1 is chosen as a constant piecewise function with one discontinuity, namely a Heavyside function. In fact, during production processes, goods are injected inside supply networks at almost constant levels in different time intervals:
$\rho_{1, b}(t, 0)=\left\{\begin{array}{cc}30, & 0 \leq t \leq \bar{t}, \\ 5, & \bar{t}<t \leq T,\end{array}\right.$
where $\bar{t}$ is the time instant at which the injection levels inside the supply network abruptly change. Notice that levels 30 and 5 of $\rho_{1, b}$ have been chosen according to the following criterion: when $0 \leq t \leq \bar{t}$, the arcs of the supply network process a great amount of goods and often reach the maximal density; when $\bar{t}<t \leq T$, the arcs process goods whose density is always less than the maximal one.

For arc 8 , we assume a boundary datum equal to $\rho_{10}{ }^{\text {max }} \square 16.667$, hence we require a possible wished output near to the maximal density processed by arc 10 .

The aim is to choose some $\bar{t}$ value, that guarantees maximal production. First we examine the behaviour of $\rho_{10}(t, x)$, for $\bar{t}=100$ and $\bar{t}=500$. The overall system is completely influenced by $\bar{t}$. In Figure 5, we notice one production peak at time, approximately, $t=400$, but the average level of density is quite low (about 0.6). Such phenomenon is not present in Figure 6, where there is one peak production, and, after it, the production decreases slowly until it reaches a fixed constant level.


Figure 5: $\rho_{10}(t, x)$ for $\bar{t}=100$


Figure 6: $\rho_{10}(t, x)$ for $\bar{t}=500$

In Figure 7 and 8, fixing $\bar{t}=500$ we show how the dynamics of the supply network is influenced by different choices of RSs at nodes 2 and 7 .


Figure 7: $\mu_{7}(t, x)$ for $\bar{t}=500$ using rule SC2 at node 2 and SC 3 at node 7


Figure 8: $\mu_{7}(t, x)$ for $\bar{t}=500$ using rule SC3 at node 2 and SC2 at node 7

The function $f\left(\rho_{10}\left(t, L_{10}\right), \mu_{10}\left(t, L_{10}\right)\right)$, namely the flux on the last point of arc 10, in the case of rules SC2 - SC3, is depicted in Figure 9 for different choices of $\bar{t}$ to understand the final product flows. The obtained results present some interesting features: first, although different values of $\bar{t}$ are used, the flux starts to be different from zero always at the same temporal instant ( $t \square 350$ ), indicating that the input flow does not influence the production dynamics, that depends only on network characteristics (initial conditions, maximal processing rates, arcs length, and so on); second, shifts of the input flow discontinuity do not foresee translations of $f\left(\rho_{10}\left(t, L_{10}\right), \mu_{10}\left(t, L_{10}\right)\right)$. Such phenomenon indicates that, also using a conservation law with a linear function and a transport equation for the production rates, the dynamics on the whole network is strongly not linear.


Figure 9: $f\left(\rho_{10}\left(t, L_{10}\right), \mu_{10}\left(t, L_{10}\right)\right)$ evaluated for different values of the discontinuity instant: $\bar{t}=100$ (continuous line), $\bar{t}=200$ (dashed line) and $\bar{t}=500$ (dot-dashed line)

The area described by $f\left(\rho_{10}\left(t, L_{10}\right), \mu_{10}\left(t, L_{10}\right)\right)$, that can have strong variations for different $\bar{t}$, represents the number of goods produced at the end of the simulation. In particular, we could ask if there exists a value $\bar{t}$ for which
$J=\int_{0}^{T} f\left(\rho_{10}\left(t, L_{10}\right), \mu_{10}\left(t, L_{10}\right)\right) d t$
is maximum.
In Figure $10, J(\bar{t})$ is reported for the following combination of rules at nodes 2 and 7: SC2 - SC2, SC2 - SC3, SC3 - SC2, and SC3 - SC3. We observe that $J(\bar{t})$ almost increases linearly for a wide range of values of $\bar{t}$ (precisely if $\bar{t} \in[200,700]$ ), until it reaches a maximum $\bar{t}^{\text {max }}$, and then it almost decreases in a constant way.


Figure 10: behaviour of $J(\bar{t})$ for different combinations of rules at nodes 2 and 7: SC2 - SC2 (dot dashed line); SC2 - SC3 (continuous line); SC3 - SC2 (dashed line); SC3 - SC3 (dot dot dashed line)

We get that $\bar{t}^{\text {max }}$ is almost insensible to rules at nodes 2 and 7 and its numerical approximation is $\bar{t}^{\text {max }} \square 830$. The just made analysis strictly depends on the input flow characteristics and network parameters. In general, the behaviour depicted in Figure 8 is a priori unpredictable due to the non linearity of supply networks, as confirmed by other similar simulation.

In Figures 11 and 12, $\rho_{10}(t, x)$ and $\mu_{10}(t, x)$ are represented for $\bar{t} \square 830$ in the case of $\mathrm{SC} 2-\mathrm{SC} 3$ rules: $\rho_{10}(t, x)$ is higher with respect to other cases already examined in Figure 5 and $6 ; \mu_{10}(t, x)$ is constant and equal to 10 . Such result is not surprising since, according to RSs at nodes, the production rates are kept equal to the initial ones on outgoing sub-chains.


Figure 11: $\rho_{10}(t, x)$ for $\bar{t} \square 830$


Figure 12: $\mu_{10}(t, x)$ for $\bar{t} \square 830$

A further remark can be done on the dependence of $J(\bar{t})$ by the distribution coefficient $\alpha$ at node 2 . In Figure 13, we represent different pictures of $J(\bar{t})$, evaluated using rules SC2 - SC3, for different values of $\alpha$. It is evident that, if $\alpha$ grows, $J(\bar{t})$ becomes higher but the value of $\bar{t}$ at which it attains its maximum point has no meaningful variations.


Figure 13: behaviour of $J(\bar{t})$ using $\mathrm{SC} 2-\mathrm{SC} 3$ for different values of $\alpha: \alpha=0.1$ (dashed line); $\alpha=0.3$ (continuous line); $\alpha=0.5$ (dot dot dashed line) and $\alpha=0.8$ (dot dashed line)

## 4. CONCLUSIONS

In this paper, starting from the model proposed in D'Apice et al. 2009, goods flows on a supply network have been studied.

An input flow of piecewise constant type with only one discontinuity has been chosen for simulating the behaviour of a supply network for chips production.

Recent studies on experimental data seems to confirm the correctness of the assumptions underlying the model. In particular, the real flow profiles on each arc are consistent with the shapes of the flux functions.

For such a network it has been proven that an accurate choice of the discontinuity point allows to maximize the total final production. The influence of the supply evolution on RSs at nodes and on the distribution parameter is analyzed.

In future we aim to develop numerical schemes to solve the optimal control problem of choosing an input flow of piecewise constant type in order to obtain an expected pre-assigned network outflow. The idea is to find the minimum of a cost functional measuring the network outflow evaluating its derivative with respect to the switching times (the controls) of the input flows through the evolution of generalized tangent vectors to the control and to the solution of the supply chain model.

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