MODELING AND SIMULATION OF PERIODIC SYSTEMS BY ISS CONTINUOUS PN

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ABSTRACT
Infinite server semantics continuous Petri nets (ISSCPNs) is one of the most relevant timed interpretation of Continuous PNs (the relaxation into continuous models of Petri nets). Previous works comparing ISSSCPNs with Forrester diagrams and linear ordinary differential equation systems (LODES), taking into account the information delays and some methodological considerations, demonstrated that ISSCPNs permit to model any LODES when known upper and lower bounds of the state variables exists. Therefore systems with cyclic behavior or delays in the information can be modeled. These results permit analyze the modeling and simulation of cyclic systems with ISSSCPNs based on the comparison with the behavior of LODES. This type of analysis is very usual in system dynamics, in order to develop models (usually with FD) for unknown systems whose evolution have been observed. The possibility of model sinusoid functions, and the addition/substation of markings in redundant places, permits ISSCPNs model any cyclical system by Fourier decomposition, and the representation of different markings leads to very interesting graphics, which can correspond to real or approximate systems. This paper deepen into ISSCPNs expressive power, that is, the type of behaviour that they can present and the kind of systems that can be modelled with them.

Keywords: Continuous Petri nets, Forrester diagrams, relaxation of discrete event dynamic systems, positive systems, expressive power.

1. INTRODUCTION

PNs constitute a well-known *family of discrete event dynamic formalism* over the nonnegative naturals. Although PNs models are originally discrete event models, their relaxation through continuization transforms them into continuous models. At the price of losing certain possibilities of analysis, this permits to obtain some advantage, such as avoiding the state explosion problem inherent to the discrete systems and taking advantage of the extensive theory about continuous dynamic systems. Although not all PN systems allow a "reasonable" continuization [1], this relaxation is possible in many practical cases, leading to a continuous-time formalism: continuous PNs. Different timed interpretations lead to different firing/flow policies. One of the most relevant is ISSCPNs, the one that will be dealt with in this paper. Under this interpretation PNs are piecewise linear systems over the nonnegative reals.

FD, a specific modelling tool inside System Dynamics (SD), provides a graphic representation of continuous dynamic systems based on (eventually non linear) ordinary differential equation systems (ODES). They have been widely used to model complex systems with a friendly graphic representation, but they are totally equivalent to ODES. An interesting class of linear ODES are positive linear systems, whose state variables take only nonnegative values, the same as Continuous PNs. Another special class of positive linear systems are compartmental systems, which are systems composed of interconnected compartments or reservoirs.

Previous works comparing ISSSCPNs with Forrester diagrams and linear ordinary differential equation systems (LODES), taking into account the information delays and some methodological considerations, demonstrated that ISSCPNs permit to model any LODES when known upper and lower bounds of the state variables exists [10].

2. PREVIOUS DEVELOPMENTS ABOUT EXPRESSIVE POWER OF ISSCPNS

The evolution of a ISSCPN is described by the system:

$$\dot{m}(\tau) = \boldsymbol{C} \cdot f(\tau)$$
$$f(\tau)[t_i] = \lambda[t_i] \cdot enab(\tau)[t_i]$$
$$\mathbf{m}(0) = \mathbf{m_0}$$

Thus a continuous Petri net under infinite servers semantics becomes a piecewise linear system. The switch between two linear systems is triggered by a change of the place giving the minimum in the expression for the enabling degree.

2.1. On positivity

Broadly speaking, positive systems are systems whose state variables take only nonnegative values. A positive system automatically preserves the non-negativity of the state variables, i.e., if non-negativity constraints on the state are added, they are redundant.

More formally, let ∑ (1) be a linear system:

$$\dot{\mathbf{x}}(t) = \boldsymbol{A}\,\mathbf{x}(t) + \boldsymbol{B}\,\mathbf{u}(t) \qquad (1)$$

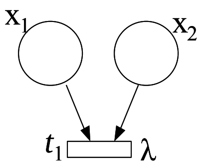
Definition 1. [9] ∑ is said to be positive iff for every nonnegative initial state and for every nonnegative input its state is nonnegative. Then the positive orthand $\Re_n^+$ is a nonnegative invariant set. If $\boldsymbol{B}=\mathbf{0}$, the system is said to be uncontrolled or unforced.

Note that positivity in linear systems can depend on the basis of the input as well as on the basis of the state space. Some non-positive system can be transformed into another equivalent positive system by a basis change in the state space. This is the reason why some authors define positive systems by requiring the existence of an invariant set (without requiring, however, that such an invariant set be the positive orthand).

Theorem 1. [9] A linear system (1) is positive, iff $\boldsymbol{A}$ is a Metzler matrix and $\boldsymbol{B}$ is nonnegative (a matrix/vector is nonnegative if all its elements are nonnegative and a square matrix is *Metzler* if non-diagonal elements are nonnegative).

According to Definition 1, ISSCPNs are positive systems (the fact that the flow of a transition is proportional to its enabling degree ensures the nonnegativity of the marking). Nevertheless, the matrices $\boldsymbol{A_i}$ of the linear systems ruling the evolution of the net (recall that an ISSCPN is a piecewise linear system) can be non Metzler matrices. In a ISSCPN the switching between linear systems is triggered by a change in the place giving the minimum in the expression of the enabling degree.

The evolution of the net system in Figure 1 is driven by a linear system with matrix A_1 if $x_1 \leq x_2$ and with matrix A_2 otherwise (if $x_1=x_2$ both systems are equivalent). Neither $\mathbf{A}_1$ nor $\mathbf{A}_2$ is a Metzler matrix, however the system is positive.



-If $x_1 \leq x_2$:
$\dot{x}_1 = -\lambda \cdot x_1$
$\dot{x}_2 = -\lambda \cdot x_1$
$\mathbf{A}_1 = \begin{bmatrix} -\lambda & 0 \\ -\lambda & 0 \end{bmatrix}$

-If $x_2 \leq x_1$:
$\dot{x}_1 = -\lambda \cdot x_2$
$\dot{x}_2 = -\lambda \cdot x_2$
$\mathbf{A}_2 = \begin{bmatrix} 0 & -\lambda \\ 0 & -\lambda \end{bmatrix}$

Figure 1: A ISSCPN whose associated linear systems have non Metzler matrices.

2.2. Control Arcs in PNs. Expressive power of ISSCPNs.

Control arcs will be introduced in this section. A control arc is defined on a couple {*place*, *transition*} and allows to model instantaneous control of the flow of the *transition* without modifying the marking of the *place*. It will be shown that by using control arcs any bounded LODES can be represented by an equivalent ISSCPN.

Let us describe how a *control arc* can be added to a ISSCPN. Consider an ISSCPN with a vector of internal speeds **λ** and incidence matrices **Pre** and **Post**. Let us assumed that the flow of a transition *t* is desired to be λ[*t*] ·m[*p*] all along the evolution of the system for a given place *p* that is not an input place of *t*. In other words, we want the flow of transition *t* to be controlled by place *p*. Recall that the flow of *t* is f[*t*]=λ[*t*]· min $_{p\in \bullet t}$ {**m**[*p*]/**Pre**[*p*,*t*]}. Therefore, in order to achieve our goal it is necessary that *p* is an input place of *t* and that it is always giving the minimum in the expression for the flow. This can be done by adding an arc going from *p* to *t* with weight k. We will asume that k is big enough to ensure that *p* always gives the minimum. If the internal speed of *t*, λ'[*t*], is made k times faster (λ'[*t*]=k·λ[*t*]) then f[*t*]=λ[*t*]·**m**[*p*]. In order to avoid that transition *t consumes* fluid from *p*, a new arc going from *t* to *p* with weight k is added to the net. This way, the flow of *t* is controlled by *p*, but the marking of *p* is not modified by the firing of *t*. Summing up, to put a control arc between {*p*,*t*} of weight k, two arcs of weight k have to be added (from *p* to *t* and from *t* to *p*), and the internal speed of *t* has to be multiplied by k.

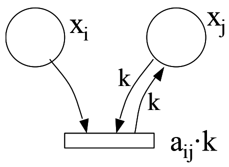


Figure 2: A control arc with weight k.

Note that in a control arc {*p*,*t*} the weight k is assumed to be big enough to ensure the control of the transition. If the markings of the input places of *t* are strictly positive and the marking of *p* is upperbounded then such a k does always exist. However, if the marking of one of the input places tends to zero or the marking of *p* tends to infinity, no finite k exists such that *p* gives the minimum in the expression for the enalbling degree.

An *ideal control arc* is defined as a control arc with its constant k equals to infinite. The use of ideal control arcs allows to control transitions even when the marking of an input place tends to zero or the marking of *p* tends to infinity. Ideal control arcs represent an extension in the modelling power of ISSCPN and can be used to empty a place in finite time. They are equivalent to the *information arcs* in FD (in linear and nonnegative restricted systems).

The following lemma states that by using regular control arcs (no ideal control arcs) any LODES that has a positive and known lower and upper bounds can be modelled by an equivalent ISSCPN.

Lemma 1: For any LODES with *positive* and known lower and upper bounds there exists an ISSCPN having identical behaviour.

Proof. (see the previous work, in reference [10])

From Lemma 1, the following Proposition is immediately obtained.

Proposition 1: For any LODES whose state variables have a known lower and upper bound there exists an ISSCPN that represents the evolution of the LODES in the positive reals.

Proof: Any LODES with known lower bounds can be shifted to the positive reals by means of a change of variable. According to Lemma 1 there exists an equivalent ISSCPN for the shifted system.

Recall that by introducing ideal control arcs it is possible for a place to control a transition even if its marking stretches to infinity. Therefore the use of ideal control arcs allows to model also those LODES that are not upper bounded.

3. ODES, FD AND ISSCPNS

3.1. Behaviours types modelled with ISSCPN

Let be a general case of unforced linear ODES, $\dot{\mathbf{x}}(t) = \boldsymbol{A}\,\mathbf{x}(t)$, with dimension 2, where:

$$A = \begin{bmatrix} a11 & a12 \\ a21 & a22 \end{bmatrix}$$

The system can be represented by FD as shown in Figure 3. By ISSCPN if can be represented according to Table 1, which uses ideal control arcs in the cases with negative coefficients.

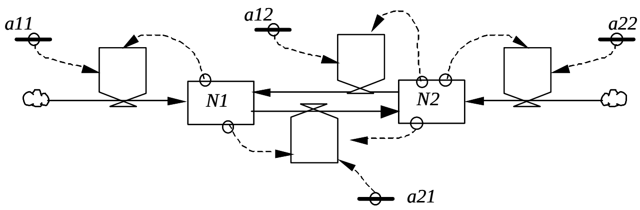
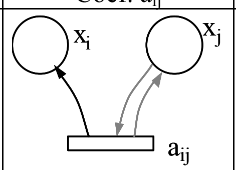
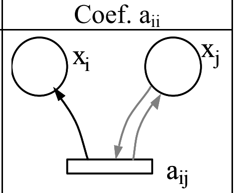
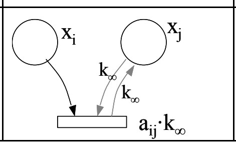
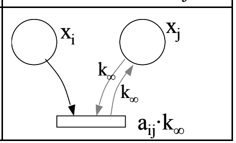


Figure 3: FD modelling a general unforced linear ODES of dimension 2.

Table 1: ISSCPN modelling general unforced linear ODES, depending on the signs of the coefficients.

	Coef. a_{ij}	Coef. a_{ii}
> 0		
< 0		

As control arcs can be used in both cases (positive or negative coefficients), the system can be represented by ISSCPN as shown in Figure 4, in which the sense of some arcs depends on the sign of the coefficient (the narrow with a + must be used with positive coefficients and viceversa).

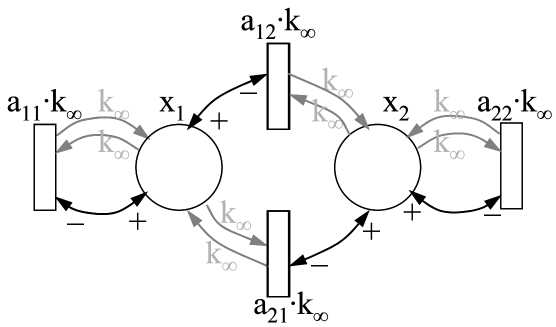


Figure 4: ISSCPN modelling a general unforced linear ODES of dimension 2.

The solution of the system is, in the general case, $e^{\mathbf{A}t} \cdot \mathbf{x}(0)$. Certain particular cases that exemplify the types of behaviour of this system are shown in Table 2. Exponential (positive and negative), linear, oscillating, hyperbolic and even sine growing behaviours can been seen.

Table 2: Examples of behaviour that can be modelled with ISSCPNs in positive systems.

A	eig(A)	e^{At}
$\begin{bmatrix} \pm a_1 & 0 \\ 0 & \pm a_2 \end{bmatrix}$	$\pm a_1, \pm a_2$	$\begin{bmatrix} e^{\pm a1\cdot t} & 0 \\ 0 & e^{\pm a2\cdot t} \end{bmatrix}$
$\begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}$	a, 0	$\begin{bmatrix} 1 & 0 \\ a\cdot t & 1 \end{bmatrix}$
$\begin{bmatrix} 0 & a \\ 1/a & 0 \end{bmatrix}$	1, –1	$\begin{bmatrix} \cosh t & a\cdot\sinh t \\ (\sinh t)/a & \cosh t \end{bmatrix}$
$\begin{bmatrix} 0 & a \\ -1/a & 0 \end{bmatrix}$	i, –i	$\begin{bmatrix} \cos t & a\cdot\sin t \\ (-\sin t)/a & \cos t \end{bmatrix}$
$\begin{bmatrix} \pm a & 0 \\ \pm b & \pm a \end{bmatrix}$	±a, ±a	$\begin{bmatrix} e^{\pm a\cdot t} & 0 \\ \pm b\cdot t\cdot e^{\pm a\cdot t} & e^{\pm a\cdot t} \end{bmatrix}$
$\begin{bmatrix} 0 & a \\ -1/a & 1 \end{bmatrix}$	$\frac{1}{2}+\frac{\sqrt{3}}{2}i, \frac{1}{2}-\frac{\sqrt{3}}{2}i$	$\frac{e^{t/2}}{\sqrt{3}}\begin{bmatrix} -\sin\frac{\sqrt{3}\cdot t}{2}+\sqrt{3}\cos\frac{\sqrt{3}\cdot t}{2} & \dots \\ -\frac{1}{a}\sin\frac{\sqrt{3}\cdot t}{2} & \dots \end{bmatrix}$
$\begin{bmatrix} \pm b & a \\ -1/a & \pm b \end{bmatrix}$	±b+i, ±b–i	$\begin{bmatrix} e^{b\cdot t}\cdot\cos t & a\cdot e^{b\cdot t}\cdot\sin t \\ (-\frac{1}{a}\cdot e^{b\cdot t}\cdot\sin t)/a & e^{b\cdot t}\cdot\cos t \end{bmatrix}$

The paper is based on the behaviour derived from row 4 (which is a particular case of the last row, with b=0). It is frequently presented in usual systems. Let us suppose a particular system with a material storage (St) and a staff of employees (E), whose observed real behaviour, by Jay W. Forrester [8], was one of the basis of system dynamics. The material is decreased due to the sales (S), which are assumed to be constant in time, and it is increased with the production (P), which is proportional to the number of employees. On the other hand, E varies with the contracting (C), which is proportional to the difference between the desired storage (DSt) and the present St. Figure 5 shows the FD that models this system.

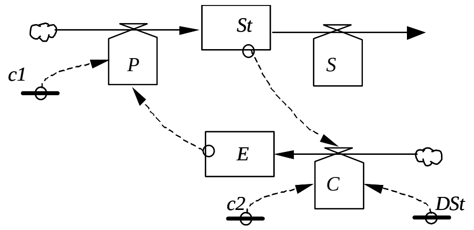


Figure 5. DF of a storage with employees system.

The differential equations system corresponding to that FD and its matrices are,

$$P(t) = c_1 \cdot E(t)$$
$$C(t) = c_2 \cdot (DSt - St(t))$$
$$dSt(t) / dt = P(t) - S$$
$$dE(t) / dt = C(t)$$

$$\boldsymbol{A}=\begin{bmatrix} 0 & c_1 \\ -c_2 & 0 \end{bmatrix} \quad \boldsymbol{B}=\begin{bmatrix} -S \\ c_2 \cdot DSt \end{bmatrix}$$

The eigenvalues of $\boldsymbol{A}$ are pure complex conjugated, independently of the values of c_1 and c_2 (due to the structure of the system), and their temporal evolution is oscillatory, sine shaped and with no damping. It can be described as:

$$St = w \cdot \sin((c_1 \cdot c_2)^{1/2} \cdot t) + DSt \quad (3)$$
$$E = w \cdot (c2/c1)^{1/2} \cdot \cos((c_1 \cdot c_2)^{1/2} \cdot t) + S/c_1$$

where *w* depends on the initial state, and it is computed as

$$w = ((St(0) - DSt)^2 + ((E(0) \cdot c_1 - S)/(c_1 \cdot c_2))^2)^{1/2}$$

So, if the storage is represented *versus* the employment, although the sales are constant a cyclic behaviour appears, with the parameters shown in Figure 6. As a curiosity, this type of behaviour (cyclic even with continuous inputs) was the origin of Forrester's studies, which were the source of System Dynamics.

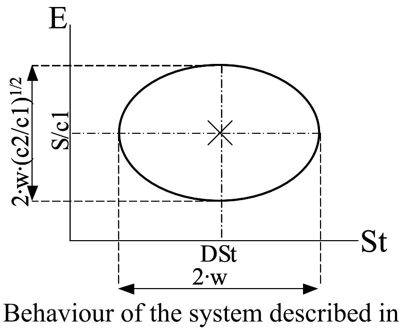


Figure 6. Behaviour of the system described in Figure 5

If St(0)=DSt and $E(0)=S/c_1$, then the system is stable. It is also important to emphasize that this system has only physical sense when the levels (stored elements and number of employees) have positive values, but the system of differential equations is non positive, and negative employment and storage can be reached for some initial conditions. Therefore, the constraint for non negativity ($St, E \geq 0$) must be additionally included in order to obtain a correct model.

The system with the non negativity constraint can be modelled with continuous PNs (Figure 7) but it must be taken into account that C, which can be positive or negative, must be implemented as a combination of a flow of new contracts and a flow of dismissals, both positive.

Note that two places have been used (those with unitary marking) in order to get a constant flow with ISS, and control arcs have been necessary to explicitly select the places that provide the information to the transitions with synchronizations. The system will be described by (3) whenever $St \geq 1/k_\infty$ and $E \geq St/k_\infty$. Recall that k_∞ represents a *finite* constant as big we want (eventually tending to infinite).

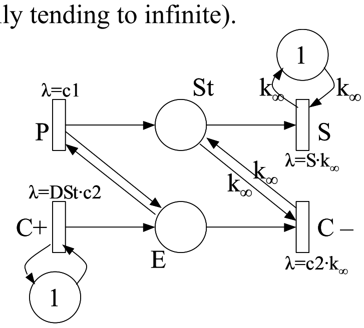


Figure 7. ISSCPN equivalent to the system in Figure 3 restricted to positive values.

It is also important to note that an appropriate value of k_∞ in the PN depends on the minimum values that St and E can reach (and then on the initial marking). For instance, if $c_1=c_2=1$, St(0)=9, DSt=10 and E(0)=S=12, then k_∞ does not need to be higher than 1. Figure 8 shows simultaneously the evolution of the constrained system (modelled with ISSCPN or FD with constraints) and the non restricted one (modelled with ODES or FD without constraints). Both are similar from the initial state to the first intersection with the horizontal axis (point *a* in the graphic). Figure 8a presents St versus E, and Figure 8b shows the temporal evolution of the state variables.

The choice of appropriate parameters can lead to completely positive systems, and then ideal control arcs are not needed. The model will be exactly the same as in Figure 7 by replacing the respective k_∞ for finite values k_i. An example of such type of evolution is represented in Figure 9.

The system can also be converted in a conservative PN if the complementary places of the state variables are added, as shown in Figure 10.

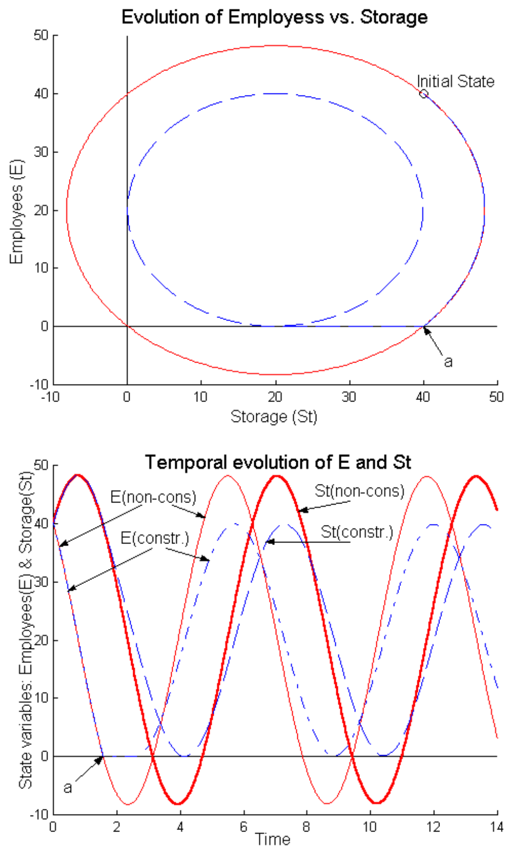


Figure 8: Evolution of the system with $c_1=c_2=1$, St(0)=E(0)=40; DSt=10, C=V=20 from constrained (broken line) and no constrained (unbroken line) models.

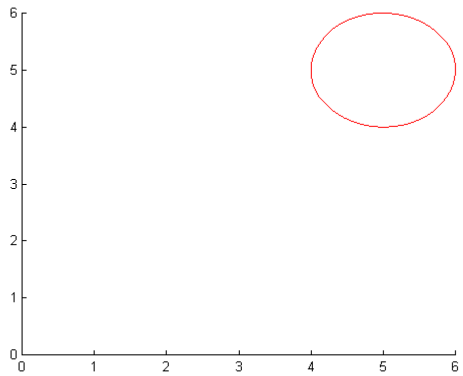


Figure 9: Evolution of the system with $c_1=c_2=1$, St(0)=4 E(0)=5; C=V=5.

Since ideal control are not used now, the system can evolution through different covertures before switching to the final coverage in which control arcs are who effectively control their transitions, as shown in Figure 11.

These coverage changes are determined by the value of the places, in the moments in which the minimum of the transitions are provided for different places, which corresponds with intersection of the curves shown in Figure 12 (intersections of red with yellow or green with blue lines).

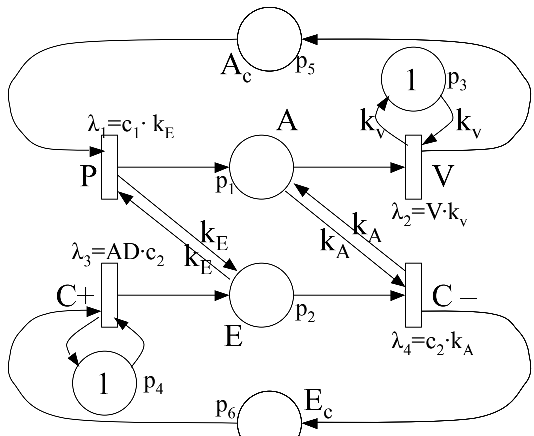


Figure 10: Conservative model equivalent to Figure 7.

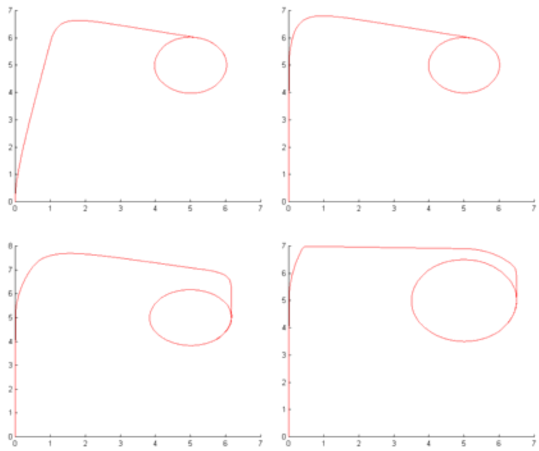


Figure 11: Different behaviours of the system depending on the parameters.

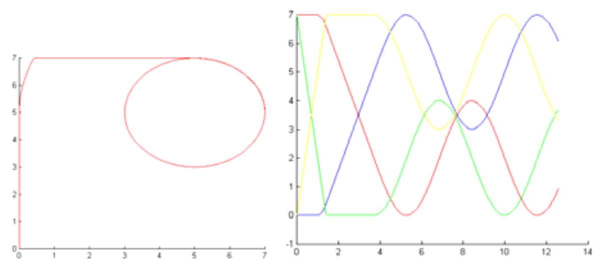


Figure 12: System evolution and temporal evolution of every variable, which permits determining the switches.

3.2. Pure systems with Petri nets

The use of control arcs occasions that the system is non-pure. A simple transformation, according to Figure 13, transforms the system into an equivalent pure PN.

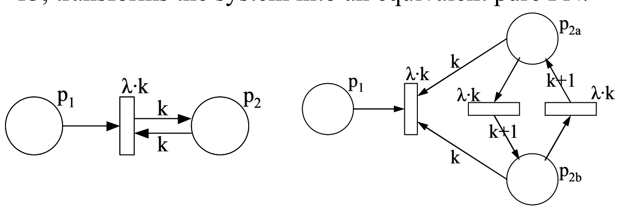


Figure 13: Control arcs modelled with pure PN.

Specifically, the example used in this paper is converted into the system in figure 14, in which the 8 arcs with weight k+1 are represented with different colour.

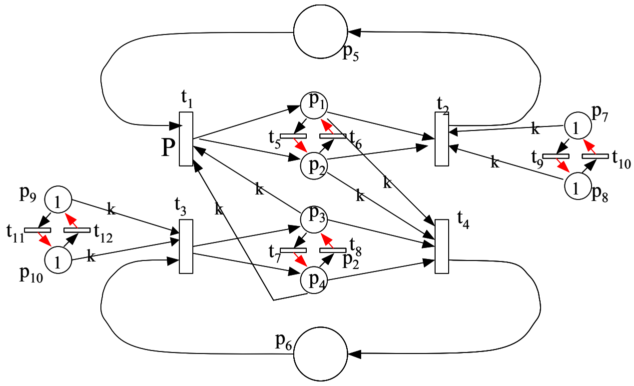


Figure 14: Conservative pure system equivalent to systems in Figures 7 and 10.

Obviously the simulation of the model in Figure 14 is exactly equal to those of the non-pure systems of Figures 7 and 10, provided that the places in which every place with control arc has been divided are exactly equals. A small difference between them can drive to stable or instable systems, as shown in Figure 15.

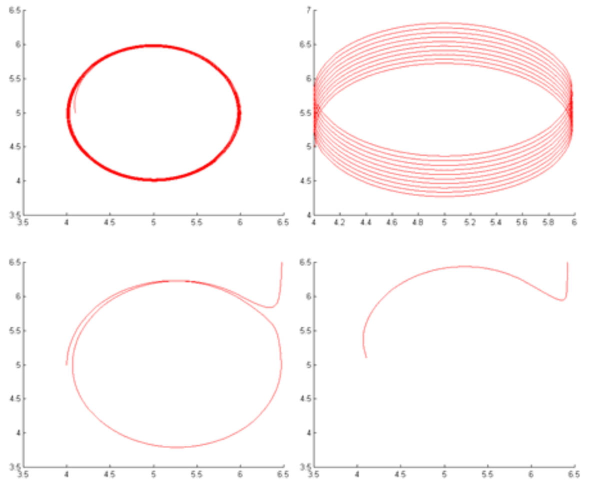


Figure 15: System evolution of model of Figure 14 with small differences in the pairs of places modelling control arcs, depending on the pairs with perturbations: a) p1-2 and p3-4, b) p7-8 c) p9-10 d) p7-8 and p9-10

4. RESULTS

A relaxed continuous view of discrete event systems, continuous Petri nets, have been considered together with Forrester Diagrams and linear ordinary differential equation systems (mainly positive systems). They have been compared in order to obtain a deeper knowledge of the expressiveness of the continuous relaxation of PNs under infinite server semantics.

Continuous Petri nets under infinite server semantics lead piecewise linear systems provided with nonnegative state and outputs (they have an internal or implicit constraint of non-negativity).

Control arcs weighted with factor *k* are an abbreviation in infinite server semantics continuous PNs, and allow to simulate bounded positive linear systems (eventually under certain transformations). But the behaviour and expressive power of infinite server semantics continuous PNs are not restricted to bounded linear positive systems. In fact, any bounded linear system (positive or not) can be shifted to the positive reals and therefore modelled by a infinite server semantics continuous PN. In particular pure oscillatory behaviours can be modelled with infinite server semantics continuous PNs.

Ideal control arcs constitute an extension for infinite server semantics continuous PNs (are the control arcs *k* weighted with factor ∞). With them its expressive power increases because they permit to simulate any positive linear system (bounded or not) or any one that can be transformed into a positive system.

The possibility of model sinusoid functions, and the addition/substation of markings in redundant places, permits ISSCPNs model any cyclical system by Fourier decomposition, and the representation of different markings leads to very interesting graphics, which can correspond to real or approximate systems.

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