ON THE INTEGRATED PRODUCTION AND PREVENTIVE MAINTENANCE PROBLEM IN MANUFACTURING SYSTEMS WITH BACKORDER

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ABSTRACT

The integrated production, inventory and preventive maintenance problem (PIPMP) is concerned with coordinating production, inventory and preventive maintenance operations in order to meet customer demand with the aim of minimizing costs. A unified framework is developed allowing production and preventive maintenance to be jointly considered using an age-dependent optimization model, itself based on the minimization of an overall cost function; this cost function for its part includes inventory holding, backlog, and preventive and corrective maintenance costs. We provide optimality conditions for more realistic manufacturing systems and use numerical methods to obtain the optimal preventive maintenance policy and the relevant age-dependent or multiple-threshold-levels production policy, which we refer to as the *multiple threshold levels hedging point policy*. Numerical examples are included to illustrate the importance and the effectiveness of the proposed methodology

Keywords: Preventive maintenance, Buffer inventory, Backorder, Reliability theory, Manufacturing Systems.

1. INTRODUCTION

Preventive maintenance involves a schedule of planned maintenance actions aimed at the prevention of breakdowns and failures. The long-term benefits of preventive maintenance include: improved system reliability, decreased replacement cost, decreased system downtime and better spares inventory management. The aim of this paper is: (i) to propose a probabilistic control model for the simultaneous planning of the production and preventive maintenance of a manufacturing system, and (ii) to develop an efficient technique for the computation of the optimal control policy considered.

An overview of relevant literature reveals that significant contributions have been proposed based on (i) preventive maintenance, (ii) production control, and (iii) joint production and maintenance optimization models. It has been shown in [10] and [11] that policies which do not address maintenance and production control decisions in an integrated manner can perform quite poorly. Available models are considered individually or simultaneously, and are restricted to simplified assumptions that sometimes lead to not-so-realistic preventive maintenance or production policies. Comparisons of long-term effects and costs usually favour preventive maintenance over the performance of maintenance actions only when the system fails. The age replacement policy (ARP) is one of the available preventive maintenance options, and takes precedence over the block replacement policy (BRP) or group replacement policy (GRP). For details on these policies, please see [3] and [1] and their corresponding references. Details on other maintenance policies and their effects on the productivity and availability of a manufacturing system can be found in [17]. With the age replacement policy, which is a basic and simple replacement policy, the unit is replaced upon failure or at a preset age, whichever occurs first [9]. A generalized age-replacement policy with age-dependent minimal repair and random lead-time is presented in [18] by considering the average cost per unit time and the stochastic behavior of the system considered. The model includes the cost of storing a spare as well as the cost of system downtime. However, the implementation of an age replacement policy requires the continuous tracking of a component's service life. This explains the popularity of the block replacement policy (BRP) in industries with large systems, each having a specific number of components. Given that ARP is based on age-dependent preventive maintenance periods rather than fixed periods, as is the case with BRP, it remains more realistic, and thus attracts many researchers. We refer the reader to extended versions of age replacement policies and their implementation presented in [1]. The related policies are non-realistic in the context of manufacturing systems, given that frequent machine breakdowns inevitably create bottlenecks for the process. Hence, preventive maintenance (used to reduce the likelihood of machine breakdowns) combined with the control of finished goods inventories, is a potential means of reducing the overall incurred cost.

The dynamics of the finished goods inventory are not considered in the aforementioned models, which are classified here as static models, given that the policies obtained are based on the mean values of the stochastic processes involved. Manufacturing systems with unreliable machines have been modeled using the stochastic optimal control theory, in which failures and repairs processes are supposed to be described by homogeneous Markovian processes. The related optimal control model falls under the category of problems presented in the pioneering work of Rishel [15] and in [4]. The analytical solution of the one-machine, one-product manufacturing system presented in [2] result from investigations carried out in the same direction. Preventive maintenance planning problems are combined with production control to increase the availability of the production system, and hence to reduce the overall incurred cost [5].

The preventive maintenance model for a production inventory system is developed in [7] using information on system conditions (such as finished product demand, inventory position, costs of repair and maintenance, etc.) and a continuous probability distribution characterizing the machine failure process. An analytical model of the BRP and safety stock strategy is formulated in [14], also using restrictive assumptions such as the fact that the time to accomplish build-up and depletion of safety stock is small relative to the mean time to failures (MTTF). The model presented in [16] combines the ARP with the safety stock to show that inventory needs to be built just before the preventive maintenance occurs. It is assumed in [16] that extra capacity is maintained in order to hedge against the uncertainties of the production processes, and that there will be no possible breakdown of the machine before the preventive maintenance date. Without the assumption made in [16] on machine dynamics, the stochastic optimal control theory is used in [6], [12] and [13] to define machine age-dependent production and preventive maintenance policies. Such policies are based on increasing failure rate (IFR) distributions, and are characterized by a staircase structure, but with only one step.

The purpose of this paper is to investigate the joint implementation of preventive maintenance and safety stocks in a manufacturing environment in the presence of more realistic features than those made in [14] and [16]. The model presented in this paper is applied to a manufacturing system capable of catching up with unmet demand without interrupting the normal production process, as soon as production resumes (backorder situation). We consider the possibility of having a breakdown during the catch-up period. Previous models in the literature assumed that such a possibility is negligible. In addition, there is no restriction on any of the operational, repair and preventive maintenance time distributions (i.e., there is no restriction regarding the exact type of distribution of the time to machine breakdown, the time to corrective maintenance and the time to preventive maintenance).

The optimal production and preventive maintenance policies obtained in this paper significantly reduce the incurred cost, and are shown to be characterized by a multiple-step staircase structure describing the fact that the stock threshold level increases with the machine age given that its failure probability also increases with age. Significant stock levels at high machine ages are used to hedge against more frequent failures that occur randomly in such situations. The staircase structure of the control policy is shown to be the major contribution of the proposed model. The performance of such a policy is compared to that of the single stock threshold level control policy presented in [2] and extended in [12]. The model presented here is developed through the characterization of the stochastic processes underlying the system. Using the properties of the probability structure of these stochastic processes, the overall incurred cost is considered as a performance criterion, and is used as a basis for optimally determining the set of age-dependent stock threshold levels for each operational time or for each range of machine ages.

2. MODEL NOTATIONS AND ASSUMPTIONS

Throughout the article, we will be using the following notations and assumptions:

Notations

ARP Age Replacement Policy
GFR General Failure Rate
HPP Hedging Point Policy
IFR Increasing Failure Rate
DFR Decreasing Failure Rate
MADP Multiple Age-Dependent Policy
MTTF Mean Time to Failures
MTTR Mean Time to Repair
SADP Single Age-Dependent Policy
SIFRR Single Increasing Failure Rate Region
c^+ Inventory holding cost per unit time
c^- Penalty cost for each unit of unmet demand
c_1 Cost of corrective maintenance
c_2 Cost of preventive maintenance
$u(\cdot)$ Production rate of the system
$u_{\max}$ Maximal production rate
d Demand rate
$\zeta(t)$ State of the machine at time *t*
T_b The time to machine breakdown
$f(t)$ Probability density function of T_b
$F(t)$ Cumulative distribution function of T_b
$R(t)$ Reliability function of the machine
μ Mean time to machine breakdown
T_{pm} Time to preventive maintenance
$q(t)$ Probability density function of T_{pm}
T_{cm} Corrective maintenance time
$g(t)$ Probability density function of T_{cm}
S Single stage stock threshold level

S_i Stage *j* stock threshold level or stock level

dA_j interval j of the machine age partition

T Scheduled time to preventive maintenance

Assumptions

The fundamental assumption of ARP, that the cost of failure replacement (c_1) is greater than the cost of preventive replacement (c_2), is also applicable here. Other assumptions considered in this paper are:

1. All failures are instantly detected and repaired.
2. If a machine failure occurs during a production phase, corrective repair is started immediately and after repair, the machine is restored back to the same initial working condition. In addition, a preventive maintenance action (as a corrective one) renews the production system (i.e., the age of the machine is set to zero).
3. The mean value of the time requirement for a preventive maintenance operation is short when compared with the mean time to machine breakdown (i.e., $E[T_{pm}] < E[T_b] = \mu$).
4. A sufficient capacity is present to allow the accumulation of safety stock at the beginning of each machine life cycle.
5. The time to accomplish the build-up and depletion of safety stocks is not necessarily small relative to the MTTF (i.e., a breakdown could arise during that time) and the system is feasible (i.e., $\frac{MTBF}{MTBF+MTTR}\cdot u_{\max} \geq d$)
6. All unmet demand is not lost, as is the case in many others works in the relevant literature.

3. PROBLEM STATEMENT

The manufacturing system considered consists of a single machine which is subject to random breakdowns and repairs. The machine in question can produce one part type and its state can be classified as operational, denoted by 1, under repair, denoted by 2, and under preventive maintenance, denoted by 3. Assume that $\zeta(t)$ denotes the state of the machine with value $M = \{1, 2, 3\}$.

The dynamics of the machine could be described by a stochastic process, with jumps from modes α to mode β, illustrated in Figure 1 by Jump($\alpha \to \beta$). Figure 1, called a transition diagram, describes the dynamics of the machine considered.

The transition from mode (operational mode) to mode 2 (repair mode) is machine age-dependent, and is described by an increasing failure rate (IFR) distribution. The transition from mode 1 to mode 3 (from operational to preventive maintenance modes) is controlled in order to increase the capacity of the considered system characterized by an IFR distribution and an overall cost of maintenance which is lower than the overall cost of a corrective maintenance.

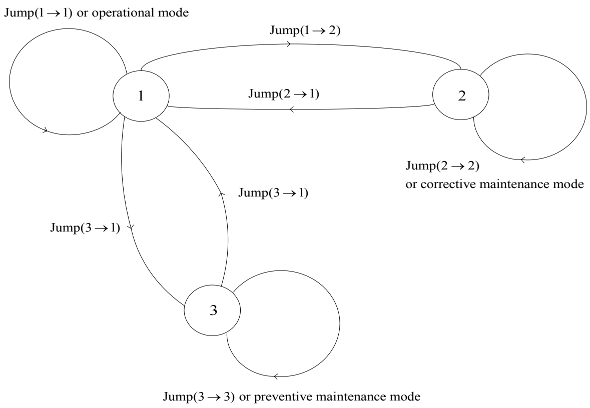


Figure 1: States transition diagram of system

Based on the costs ratios and inventory level, the proposed model will determine an optimum time for a preventive maintenance action. In a more realistic manufacturing context (no restriction on any of the operational, repair and preventive maintenance time distributions, as mentioned in the Assumptions section), three different distributions are considered in this paper, rather than using a Markov process, as in [8] and references therein.

The system behavior is described by a hybrid state comprising both a discrete and a continuous component. The discrete component consists of the discrete event stochastic processes describing modes 1, 2, and 3. Let us assume that $u(\cdot)$ denotes the production rate of the machine at time t for a given stock level x and time for preventive maintenance T. The feasible production policies set is given by

$$\Gamma = \left\{ \left(u(\cdot) \right) \in \Re, 0 \leq u(\cdot) \leq u_{\max} \right\} \tag{1}$$

where $u(\cdot)$ is known as the control variable, and constitutes the so-called control policy of the problem under study. The continuous component consists of a continuous variable $x(\cdot)$ corresponding to the inventory/backlog of products. This state variable is described by the following differential equation:

$$\frac{dx(\cdot)}{dt} = u(\cdot) - d \qquad x(0) = x \tag{2}$$

where *x,* and *d* are the given initial stock level and demand rate respectively. Let $g(\cdot)$ be the cost rate defined as follows:

$$g\left(\alpha(t), x, \cdot\right) = c^+x^+ + c^-x^- + c_1\text{Ind}\{\alpha(t)=2\} + c_2\text{Ind}\{\alpha(t)=3\} \tag{3}$$

where c^+ and c^- are costs incurred per unit of produced parts for inventory and backlog respectively, $x^+ = \max(0, x)$, $x^- = \max(-x, 0)$ and

$$\text{Ind}\{\Theta(\cdot)\} = \begin{cases} 1 & \text{if } \Theta(\cdot) \text{ is true} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

for a given proposition $\Theta(\cdot)$. The corrective and preventive maintenance activities involve constant costs, namely c_1 and c_2 respectively.

Our objective is to control the production rate $u(\cdot)$ so as to minimize the overall total cost, integrating the instantaneous cost given by equation (3).

Whenever a breakdown occurs, corrective maintenance is performed, during a random amount of time, in order to restore the machine to its initial condition (i.e., the machine is assumed to be new, and its age is reset to zero). During the maintenance period, one of the following two situations occurs:

- demands for items are met only through safety sock;
- unmet demands for items are backlogged.

In order to reduce the likelihood of machine breakdown, preventive maintenance is scheduled and combined with production planning such that each time, immediately after the maintenance operation is performed, the machine is restored to its initial working condition.

The machine state moves from modes 1, 2 and 3 according to random variables T_b , T_{cm} and T_{pm} defined as time to machine breakdown, corrective maintenance time and preventive maintenance time. At mode 1 and for given age a and scheduled preventive maintenance period T , the production rate is given by an extended version of the so-called hedging point policy (HPP), defined by a threshold level, which is valid just for the value of the age, called here $S(t)$. Such a policy is given by:

$$u(x)=\begin{cases} u_{\max} & \text{if } \ x<S(t) \\ d & \text{if } \ x=S(t) \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

An example of a surplus trajectory is illustrated in Figure 2, for which we assumed a constant threshold $S_{dt}(t)$ for an infinitesimal index dt at age t (i.e., $\lim_{dt\to 0} S_{dt}(t)=S$). The holding cost in such a situation reflects the average inventory held over the period t to $t+dt$. For a given initial inventory and a given final inventory, the average of the two is taken to be the average inventory. In the cases of manufacturing systems considered here, demand is time-homogeneous, at least in expectation, and so this assumption is valid. The demand rate of the product is a known constant whereas the production rate (which is greater than the demand rate) depends on the decision variables $S(t)$ and T. For the seek of simplicity, we use S instead of $S(t)$ in the rest of the paper without loosing the fact that S is age dependent. During production, the machine and surplus dynamic both involve the four scenarios presented in the next section.

4. OPTIMALITY CONDITIONS

Scenario No. 1. There is a breakdown before the scheduled preventive maintenance time, and the repair process involved ends with inventory or at a positive surplus level. The finished goods inventory in such a situation is illustrated in Figure 2. The holding cost in the period [0.T] is computed using area A1-1 illustrated in Figure 2.

$$\text{A1-1} = S\cdot\int_0^T R(t)dt$$

where $R(t)$ is the reliability function of the machine and t_f , as represented in figure 2, is its failure age with $t_f < T$. The holding cost, for scenario 1, is thus given by:

$$\begin{aligned} Cost_{1-1} &= c^+\cdot \text{A1-1} \\ &= c^+\cdot S\cdot\int_0^T R(t)dt \end{aligned} \quad (6)$$

The repair process ends at time t_r before the zero inventory point. The subsequent production cycle is undertaken and restarted at zero inventory, as illustrated in Figure 2. This is obtained by letting $s(a):=0$ in the control policy given by equation (5) until t_0 where $x(t)=0$. From the failure time t_f to the beginning of the subsequent production cycle t_0 , the holding cost involved is computed using area A1-2 illustrated in Figure 2.

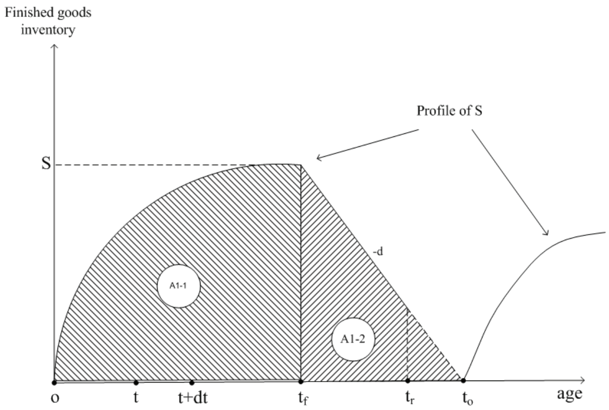


Figure 2: Inventory sample path in the case of failure before preventive maintenance without backlog

The area A1-2 is given by the following expression:

$$\text{A1-2} = \int_0^T \left[\frac{S^2}{2d}\right] dF(t)$$

where $dF(t)$ is the probability of a breakdown occurrence in interval $[t, t+dt]$. The holding cost, for scenario 2, from t_f to t_0, is thus given by:

$$\begin{aligned} Cost_{1-2} &= c^+\cdot \text{A1-2} \\ &= c^+\cdot\int_0^T \left[\frac{S^2}{2d}\right] dF(t) \end{aligned} \quad (7)$$

The cost incurred in periods $[0,t_f]$ and $[t_f,t_0]$, for scenario 1, is determined using equations (6) and (7). Such a cost is given by:

$$\begin{aligned} Cost_1 &= Cost_{1-1} + Cost_{1-2} + c_1 \cdot F(T) \\ &= c^+ \cdot S \cdot \int_0^T R(t)dt + c^+ \cdot \int_t^{t+dt} \left[\frac{S^2}{2d}\right] dF(t) + c_1 \cdot F(T) \end{aligned} \tag{8}$$

Scenario No. 2. There is a breakdown before the scheduled preventive maintenance time, and the repair process ends with a backlog situation or at a negative surplus level. The finished goods inventory in such a situation is illustrated in Figure 3. The holding cost in scenario 2 is given by equation (8), and the backlog cost is described by proposition 1.

Proposition 1. By multiplying the number of backlogged unmet demand units by a penalty cost, we obtain the following backlog cost for scenario 2.

$$Cost_2 = c^- \cdot A2 = c^- \cdot \int_0^T \left[\left(\int_{\frac{s}{d}}^{\infty} \left(x - \frac{S}{d} \right) \cdot g(x)dx \right)^2 \cdot \frac{d}{2} \cdot \left(\frac{u_{\max}}{u_{\max} - d} \right) + c_1 \right] dF(t) \tag{9}$$

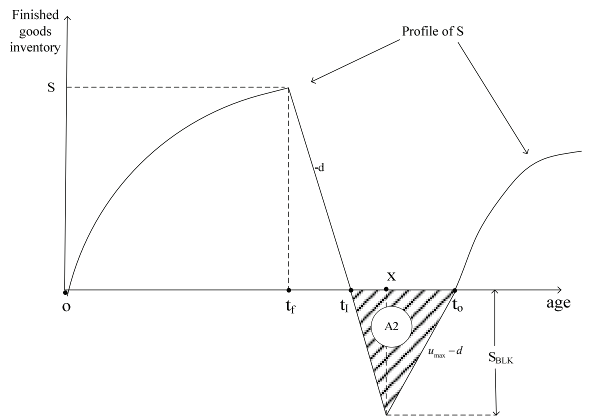


Figure 3: Inventory sample path in the case of failure before preventive maintenance with backlog

Proof: From Figure 3, we can write the following expressions:

$$t_l - t_f = \frac{S}{d} \tag{10}$$

$$x - t_l = \int_{\frac{s}{d}}^{\infty} \left(x - \frac{S(t)}{d} \right) \cdot g(x)dx \tag{11}$$

$$\begin{aligned} S_{BLK} &= d \cdot (x - t_l) \\ &= (t_0 - x) \cdot (u_{\max} - d) \end{aligned} \tag{12}$$

Using equations (10), (11) and (12), the time needed to produce parts for backlogged demand units (i.e., from x to t_0) is given by:

$$t_0 - x = \frac{d}{u_{\max} - d} \cdot \int_{\frac{s}{d}}^{\infty} \left(x - \frac{S(t)}{d} \right) \cdot g(x)dx \tag{13}$$

From Figure 3 (see area $A2$), $t_0 - t_l = (t_0 - x) + (x - t_l)$. Equations (11) and (13) give the following expression:

$$t_0 - t_l = \frac{u_{\max}}{u_{\max} - d} \cdot \int_{\frac{s}{d}}^{\infty} \left(x - \frac{S(t)}{d} \right) \cdot g(x)dx \tag{14}$$

Area $A2$ is then given by:

$$\begin{aligned} A2 &= \frac{S_{BLK} \cdot (t_0 - t_l)}{2} \\ &= \frac{d \cdot u_{\max}}{2(u_{\max} - d)} \cdot \left(\int_{\frac{s}{d}}^{\infty} \left(x - \frac{S(t)}{d} \right) \cdot g(x)dx \right)^2 \end{aligned} \tag{15}$$

Finally, the backlog cost in scenario 2 is

$$Cost_2 = c^- \cdot \int_0^T A2dF(t) + c_1 F(T) \tag{16}$$

For scenarios 1 and 2, a corrective maintenance cost is added at the end of the cycle, given that the machine fails before the scheduled preventive maintenance.

Scenario No. 3. There is no breakdown before the scheduled preventive maintenance time, and the maintenance process ends with an inventory or positive surplus level.

A preventive maintenance is performed on the machine at the scheduled time T (Figure 4), and is completed at $T + T_{pm}$ as illustrated in Figure 4.

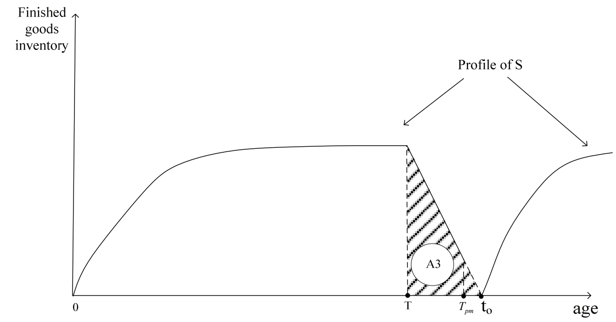


Figure 4: Inventory sample path in the case of no failure before preventive maintenance without backlog

For this scenario, there is no failure before T and the holding cost is computed as in scenario 1, evaluated at T (i.e., for $t = T$), and added to the preventive maintenance cost. The cost is then given by:

$$Cost_3 = c^+ \cdot A3 = c^+ \cdot S \cdot \left(\frac{S^2}{2d} + \int_0^T R(t)dt \right) + c_2 \cdot R(T) \tag{16}$$

Scenario No. 4. There is no breakdown before the scheduled preventive maintenance time, and the maintenance process ends with a backlog or negative surplus level.

A preventive maintenance is performed on the machine at the scheduled time T and is completed at $x = T + T_{pm}$ as illustrated in Figure 5. The holding cost is computed as previously, evaluated at T (i.e., for $t = T$). The backlog cost for scenario 4 is:

$$Cost_4 = c^- \cdot A4 = c^- \cdot \frac{d \cdot u_{\max}}{2(u_{\max} - d)} \cdot \left(\int_{\frac{s}{d}}^{\infty} \left(x - \frac{S}{d} \right) \cdot q(x)dx \right)^2 + c_2 \cdot R(T) \tag{17}$$

We may recall that the machine age is reset to zero after each operation on the machine (corrective or preventive maintenance).

Using expressions developed through scenarios 1 to 4, the overall cost $L(S,T)$, for one production cycle, ended after a maintenance action, depending on variables S and T, is given by the following expression:

$$C(S,T)=\int_0^T\left[c^+\cdot S\cdot R(t)\cdot dt\right.$$
$$+\int_0^T\left[c^+\cdot\frac{S^2}{2d}+c^-\cdot\left(\int_{\frac{s}{d}}^{\infty}\left(x-\frac{S}{d}\right)\cdot g(x)dx\right)^2\cdot\frac{d}{2}\cdot\left(\frac{u_{\max}}{(u_{\max}-d)}\right)\right]\Bigg]\cdot dF(t)+c_1F(T)$$
$$+c^-\cdot\frac{d\cdot u_{\max}}{2(u_{\max}-d)}\cdot\left(\int_{\frac{s}{d}}^{\infty}\left(x-\frac{S}{d}\right)\cdot q(x)dx\right)^2+c_2R(T) \tag{18}$$

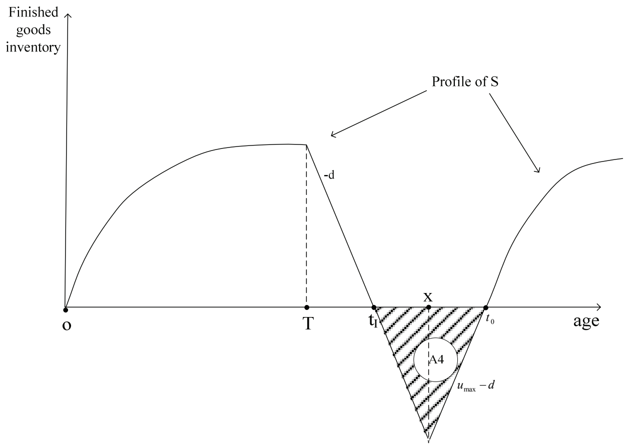


Figure 5: Inventory sample path in the case of no failure before preventive maintenance with backlog

According to According to assumption 2, the duration of a production cycle is approximated given by

$$Cycle\_Time=m(T)+R(T)\cdot MTTP+F(T)\cdot MTTR \tag{19}$$

where The mean operational time of the machine is described by $m(T)$ given by:

$$m(T)=\int_0^T t\cdot f(t)\cdot dt+R(T)\cdot T \tag{20}$$

By dividing $C(S,T)$ by the production cycle time, one obtains the overall expected cost per unit time $L(\cdot)$, given by the following equation:

$$L(S,T)=\frac{C(S,T)}{m(T)+R(T)\cdot MTTP+F(T)\cdot MTTR} \tag{21}$$

In previous scenarios, in order to develop the expression of the overall cost given by equation (16), we assumed the following behavior for the stock:

- The surplus is null at the beginning of the production period, just after an operation on the machine (corrective or preventive maintenance) or at the beginning of the production horizon;
- After an operation on the machine, if the surplus is positive, the production process needs to be stopped and reset to zero inventory using the production policy given by equation (5), and updated here using the aforementioned considerations.

Let us now develop the mathematical model and the optimality conditions to determine values of optimal stock level and maintenance period $S^*(a)$ and $T^*(a)$ for machine age a.

Proposition 2. The function $L(S,T)$ given by equation (16) is convex in S and T.

Proof. Since the holding and backlog cost function is convex and the convexity is maintained under expectation (cf. integration), the proposition follows.

Proposition 2 and previous scenarios are used to show that the optimality conditions, and hence, the age-dependent optimal values of the control policy (threshold levels and scheduled preventive maintenance times), are given by Theorem 1.

Theorem 1. A safety stock S^* and a scheduled preventive maintenance time T^* are optimal at time t if and only if:

$$\frac{\partial L(S,T)}{\partial T}=0 \quad \text{and} \quad \frac{\partial L(S,T)}{\partial S}=0$$

For the safety stock, we have:

$$\frac{\partial L(\cdot)}{\partial S}=c^+\cdot\int_t^{t+\Delta t}R(t)\cdot dt+\Delta F(t)\cdot\left[c^+\cdot\frac{S}{d}+c^-\cdot\left[-\frac{S}{d}\cdot g\left(\frac{S}{d}\right)-\frac{q\left(\frac{S}{d}\right)}{d}+\frac{S}{d}\cdot g\left(\frac{S}{d}\right)\right]\times\right.$$
$$\left.\left[\left(\int_{\frac{s}{d}}^{\infty}\left(x-\frac{s}{d}\right)\cdot g(x)\cdot dx\right)\cdot\frac{d}{2}\cdot\left(1+\frac{d}{(U_{mx}-d)}\right)\right]\right]=0 \tag{22}$$

and certainly,

$$S-2\cdot\frac{c^-}{c^+}\cdot F\left(\frac{s}{d}\right)\cdot\left[\int_{\frac{s}{d}}^{\infty}\left(x-\frac{s}{d}\right)\cdot g(x)\cdot dx\right]\cdot\frac{d}{2}\cdot\left(1+\frac{d}{(U_{mx}-d)}\right)=-\frac{d\cdot\int_t^{t+\Delta t}R(t)\cdot dt}{\Delta F(t)} \tag{23}$$

Proof. In the interval $[0,T]$, the first-order sufficient condition to obtain the threshold value S is $\frac{\partial L(\cdot)}{\partial S}=0$ with function $L(\cdot)$ described by equation (21). Equation (22) is obtained, and the threshold value or optimal safety stock in $[0,T]$ is given by equation (23). Due to the complexity of the expression (21) with respect to T, obtaining the analytical expression for such a parameter, as in the case of the parameter S, becomes more complex. Hence, we use the convexity property of the function $L(\cdot)$ to complete the proof of this theorem for the parameter T.

We solve equation (23) for S and use a numerical search over a given computational grid to provide the minimal cost and the related control parameters (S for production policy and T for preventive maintenance policy).

5. NUMERICAL PROCEDURE

The time to system failure T_b has a Weibull (β, η) distribution. Here, β and η are the shape and scale (characteristic life) parameters of the distribution. We must recall that a Weibull distribution is IFR when $\eta \geq 1$, and decreasing failure rate (DFR) when $0 < \eta \leq 1$; when $\eta = 1$, $h(t) = 1/\eta = \lambda$, we obtain the exponential distribution which is both IFR and DFR. In this paper, we consider a *Rayleigh* distribution, which is a special case of Weibull, with $\eta = 2$ (i.e., IFR with $h(\infty) = \infty$).

The time to corrective maintenance T_{cm} and the time to preventive maintenance T_{pm} have a Lognormal (μ, σ) distribution, where the two parameters, μ and σ, are the mean and standard deviation of the natural logarithm of the time to perform the operation, corrective or preventive maintenance, represented here by Lognormal random variables. The Lognormal distribution is a single increasing failure rate region (SIFRR), which is a special class of general failure rate (GFR), given that it only has one IFR region ($h(\infty) = 0$).

Given the previous probability distribution functions, the values of $L(S,T)$ were derived from numerical integration methods (Theorem 1) and from the proposed algorithm, to obtain optimal values of $L(\cdot)$, S and T (denoted as $L^*(\cdot)$, S^* and T^* henceforth). An algorithm for the optimal production and preventive maintenance planning problem is given in Figure 6, and was coded using the Matlab/Simulink software.

Based on the proposed algorithm, the numerical scheme proceeds as follows: (i) read input data; (ii) consider computational grid on T for given lower and upper bounds $T^{\min}$ and $T^{\max}$ respectively; (iii) for each feasible scheduled preventive maintenance time T (i.e., $T^{\min} \leq T \leq T^{\max}$), consider a discrete time interval dt and solve the optimality condition at time t to obtain the optimal cost and the associated threshold level.

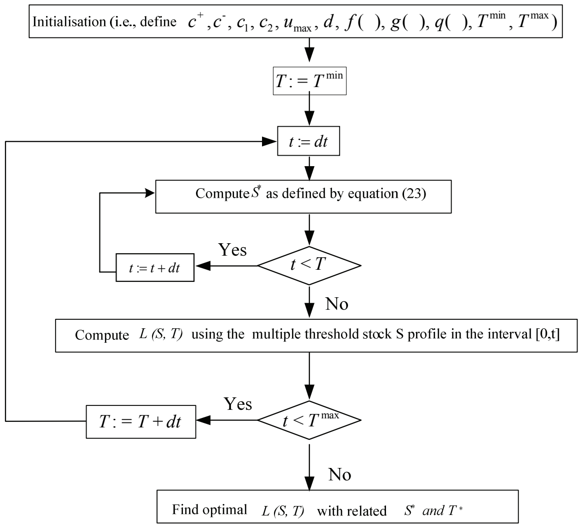


Figure 6: Iterative numerical procedure or algorithm

For illustration purposes, assume $u_{\max} = 1$ item per unit of time and the production process is run to satisfy a constant demand rate $d = 0.5$ item per unit of time. The time to breakdown T_b is Weibull, with $\beta = 2$ and $\eta = 100$ (i.e., $\mu = 88.6$). The time to repair and the time to preventive maintenance are Lognormal, with $\mu_{cm} = 20$, $\sigma_{cm} = 1$ and $\mu_{pm} = 10$, $\sigma_{pm} = 0.5$ respectively. For such a system, a sensitivity analysis is provided in the next section to show the usefulness of the proposed approach and to illustrate the contribution of this paper.

6. RESULTS ANALYSIS

Four classes of studies are considered by using the variations of inventory, backlog, corrective and preventive maintenance costs. For the first class, Figure 7 represents the effect of the holding cost variation on the stock level and the preventive maintenance time. For classes 2, 3 and 4, we illustrate the sensitivity analysis through Figures 8 to 10.

It is interesting to note the following from Figure 7, obtained with the variation of the inventory cost c^+ ($c^+ = 2, 5, 7$ and $c^+ = 1, \cdots, 45$ as in figure 7(a) and 7(b) respectively) and $c_1 = 5000$, $c_2 = 3000$, $c^- = 50$:

- The length of the first age interval increases with an increase in the inventory cost (i.e., $dA_1 = 0$ for $c^+ = 2$, $dA_1 = 18$ for $c^+ = 5$ and $dA_1 = 24$ for $c^+ = 7$). This reduces the surplus at small age values, and hence reduces the overall incurred cost (see Figure 7(a)).
- The scheduled production time for preventive maintenance T^* decreases with the increase in the inventory cost due to the fact that the safety stock decreases, and hence increases the

possibility of having a backlog situation (see Figure 7(b)).

It is clear from Figure 7 that the inventory policy significantly influences the overall incurred cost and that the safety stock increases with the machine age for each value of c^+ but with a different trend. The preventive maintenance policy is also significantly affected by the holding cost according to a staircase trend.

For the variation of the backlog cost, Figure 8(a) shows that the stock threshold levels increase with the backlog costs for given machine age intervals. In addition, the length of the first age interval (i.e., dA_1) decreases with the increase in the backlog cost. This increases the surplus at small age values, and hence increases the overall incurred cost, but the occurrence of backlog situations is minimized. It can be noted from Figure 8(b) that the preventive maintenance frequency also increases with the increase in backlog costs, thus increasing the availability of the production process, and hence the avoidance of backlog situations.

The variation of the corrective maintenance affects the preventive maintenance policy, and hence the production policy, as illustrated in Figure 9(a). The preventive maintenance frequency increases with the increase in corrective maintenance costs, as illustrated in Figure 9(b). This is done in order to reduce the number of machine breakdowns, and hence avoid corrective maintenance, which involve excessive costs. The variation of the preventive maintenance cost affects the preventive maintenance policy asymptotically (i.e., constant preventive maintenance for large costs) and hence, the production policy (operation duration) as shown in Figures 10(b) and 10(a), respectively. The results obtained show that such a variation has no significant effect on the staircase trend of the production policy (see Figure 10(a)), as in the case of corrective maintenance cost variations compared to the staircase trends illustrated in Figures 7(a) and 7(a). The preventive maintenance frequency also decreases with an increase in preventive maintenance costs, avoiding frequent preventive maintenance action, and hence reducing the total incurred cost.

7. OPTIMAL CONTROL POLICY

The results obtained, based on previous sensitivity analyses, are satisfactory and practical. In this paper (i.e., backlog case), we compare the results obtained, based on the multiple stage age-dependent control policy, to those presented in. [12], which are based on a single stock age-dependent threshold value. Table 1 shows that the proposed control policy performs well compared to a single threshold-based control policy. The cost reduction given by the proposed control policy ranges between 17% and 37% for different situations generated through a sensitivity analysis. For illustrative purposes, we include a few cases, rather than presenting all cases considered.

To show that there exists an asymptotic trend of variation of stock threshold values, we consider a situation without preventive maintenance (i.e., large value of scheduled preventive maintenance time or $T \to \infty$) and obtain the result presented in Figure 11.

A constant and large threshold level needs to be maintained when the machine age is advanced (i.e., the machine is supposed to be old), in which case the down probability is likely to be 1. The age-dependent production policy obtained, called here the MADP, is an extension of the modified hedging point policy presented in [6] and [12]. The production rate associated with the proposed MADP is given by:

$$u_j(x, age) = \begin{cases} u_{max} & \text{si } x < S_j \text{ and } age \in dA_j \\ d & \text{si } x = S_j \text{ and } age \in dA_j\,, \quad j = 1, \ldots, k \\ 0 & \text{si } x > S_j \text{ and } age \in dA_j \end{cases}$$

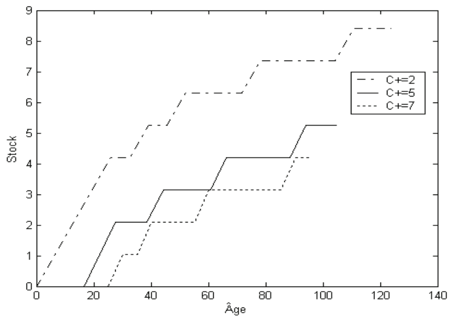
(19)

To illustrate the structure of the MADP, we consider $c_1 = 5000$, $c_2 = 3000$, $c^- = 50$, $c^+ = 2$ and obtain optimal values of S_j and dA_j. The optimal control policy to be applied to the considered manufacturing firm is defined by Equation (13) with the obtained values of S_i^* and dA_i^*, $i = 1, \ldots, 9$. The results presented in this paper indicate that as expected, the optimal production policy for the considered manufacturing system is characterized by two types of parameters, namely, optimal threshold level S_j^* and dA_i^*, $i = 1, \ldots,$ k such that the scheduled preventive maintenance period T^* is given by:

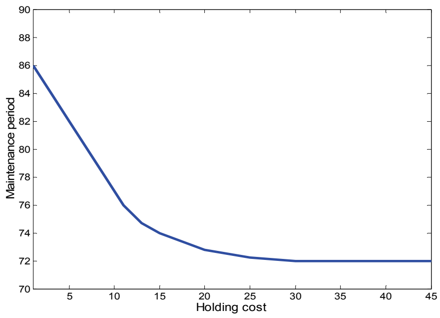
$$T = \sum_{j=1}^{k} dA_j$$

Where k is a random value obtained from the application of the algorithm presented in Figure 6. The control policy (19) is then completely defined by the values of S_j, dA_j and k.

The trends of the curves shown in Figures 7 to 11 confirm the robustness of the proposed approach through a sensitivity analysis. This is performed by threshold levels and scheduled preventive maintenance periods versus an overall incurred cost including inventory, backlog, corrective and preventive costs. The asymptotic behaviour, which is well illustrated in Figures 7 to 11, clearly shows that the results obtained make sense and that the proposed approach is robust.

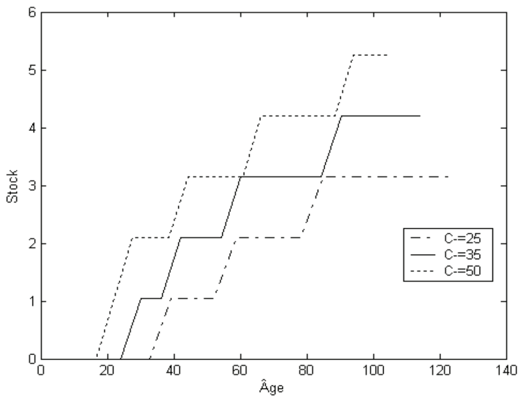


(a)

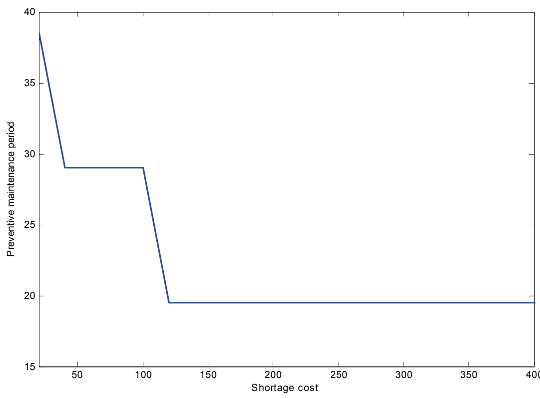


(b)

Figure 7: Optimal production and preventive maintenance policies for different inventory costs

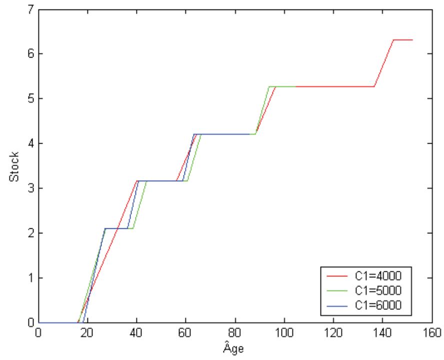


(a)

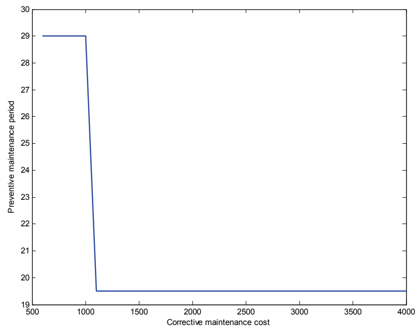


(b)

Figure 8: Optimal production and preventive maintenance policies for different backlog costs

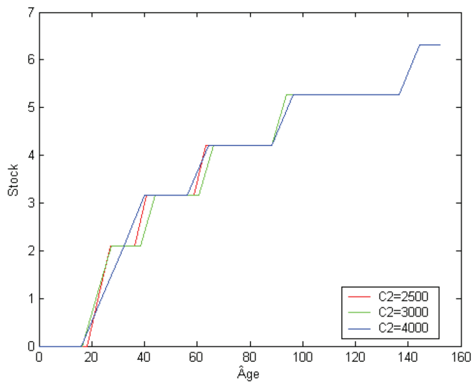


(a)

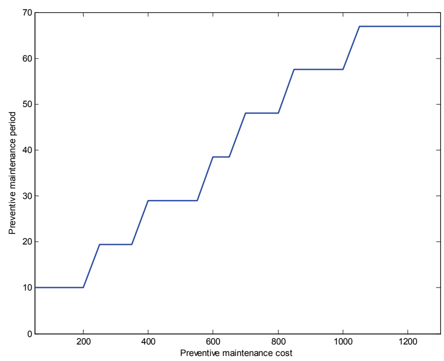


(b)

Figure 9: Optimal production and preventive maintenance policies for different corrective maintenance costs



(a)



(b)

Figure 10: Optimal production and preventive maintenance policies for different preventive maintenance costs

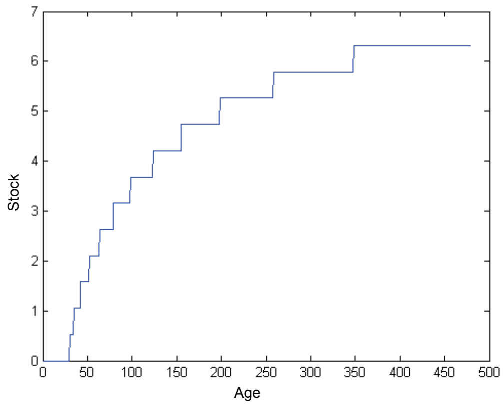


Figure 11: Multiple stage age-dependent hedging point policy with $T \to \infty$

8. CONCLUSION

A production inventory and preventive maintenance system with general characteristics and realistic assumptions has been considered here. From the results obtained, the modeling approach provides a useful tool for studying the effects of various system parameters (inventory, backlog and maintenance costs) on the overall incurred cost, as outlined by the sensitivities analysis. A new concept, called here MADP, is introduced, which is an extension of the modified age-dependent hedging point policy previously proposed in the control literature. Our model can be extended to include large manufacturing systems by generalizing the control policy given by equation (19) and using a simulation and experimental design approach after a parameterization of the control policy. Details on such an approach can be found in [8].

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