A GRAPHICAL TOOL FOR THE SIMULATION OF SUPPLY CHAINS USING FLUID DYNAMIC MODELS

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ABSTRACT

This paper presents a numerical tool for the simulation of supply chains based on two different fluid dynamic models. One is based on a mixed continuum-discrete model, it means that the load dynamics are solved in a continuous way on the arcs, and at the nodes imposing the conservation of the goods density, but not of the processing rate. In fact, each arch is modelled by a system of two equations: a conservation law for the goods density, and an evolution equation for the productive capacity. According to the other model, the load dynamics are described by a conservation law, with constant processing rate, inside each supply sub-chain, and an entering queue for exceeding parts. The dynamics at a node are solved considering an ode for the queue. The realized tool allows to reproduce the state of the densities on arcs through coloured animations.

Keywords: conservation laws, supply chains, simulation.

1. INTRODUCTION

In last years, scientific communities showed a great interest for modelling the dynamics of industrial production, managed by supply chains. The study of such dynamics can become fundamental in order to reduce some unwished phenomena (bottlenecks, dead times, and so on), which can lead to heavy delays in production processes.

Several mathematical approaches have been proposed in order to model supply chains. For example, some models are discrete and based on considerations of individual parts. Other models are continuous (see Armbruster et al. 2006a, Armbruster et al. 2006b, Armbruster et al 2004, Daganzo 2003), and based on partial differential equations.

Probably, the first paper, that relies on continuous equations, was Armbruster et al. 2006a, where the authors, via a limit procedure on the number of parts and suppliers, have obtained a conservation law (see Bressan 2000, Dafermos 1999), whose flux involves either the parts density or the maximal productive capacity.

But other continuous models for supply chains have been introduced (see Bretti et al. 2007, D'Apice et al. 2006, Gottlich 2005, Gottlich 2006) due to the difficulty of finding a solution for the general equation proposed in Armbruster et al. 2006a. Also extensions on networks have been made (D'Apice et al. 2008, Helbing et al. 2004, Helbing et al. 2005).

In this paper, we focus the attention on two different continuous models for supply chains and networks (D'Apice et al. 2006, Gottlich et al. 2005) for the simulation of chains (and more complicated networks) of big dimensions.

In particular, the model introduced in D'Apice et al. 2006, briefly (DM) model, describes supply chains by continuous arcs and discrete nodes. This implies that the load dynamics are solved in a continuous way on the arcs by a conservation law for densities and a wave equation for the maximum processing rates, and at the nodes imposing the conservation of the goods density, but not of the processing rate. In fact, each arch is modelled by a system of two equations: a conservation law for the goods density, and an evolution equation for the productive capacity.

Instead, according to the model proposed in Gottlich et al. 2005, briefly (GHK) model, the load dynamics are described by a conservation law, with a constant processing rate inside each supply sub-chain. Moreover, the adoption of queues for modelling the transition of goods among arcs is proposed. Such choice allows an easy accessibility to existence and uniqueness of the solution to the whole network (Herty et al. 2007). Unlike the first model, for which a system of partial differential equations is considered, here we rely with ODEs for queues and conservation laws (PDEs) for densities on arcs.

It is evident that the described models complete each other. In fact, the second approach is more suitable when the presence of queues with buffers is fundamental to manage goods production. The mixed continuum-discrete model, on the other hand, is useful when there is the possibility to reorganize the supply chain: in particular, the productive capacity can be readapted for some contingent necessities.

Numerical schemes for the two models of supply chains have been developed to build the core of a graphical tool, that can be useful to simulate the behaviour of an assigned supply chain, in terms of parts densities on arcs. The tool is characterized by a graphical interface, which is user – friendly, since it contains a series of buttons for an easy construction of the supply network to simulate. Inside the work area, the user is able to visualize the effects of simulations through animated coloured pictures, which show the densities on the various arcs of the supply chain, and to evaluate the effects of changes in the supply chain organization. Of course, for the (GHK) model, it is possible also to see the evolution in time of queues at nodes.

The paper is organized as follows. First, we give some basics about the models for supply chains and numerical schemes useful to approximate the equations of the various models. Then, we consider the structure of the graphical tool and present some simulation results.

2. MATHEMATICAL MODEL

We describe briefly the two models for supply chains, separately. Let us consider the (GHK) model.

A supply chain (Gottlich et al. 2005, Gottlich et al. 2006) is characterized by a set of suppliers, connected each other, with the aim of processing goods, that travel along the arcs. Moreover, each supplier consists of a processor and a buffer, or queue.

Definition. A supply chain is a graph, consisting of a finite set of arcs, *J*, and a finite set of vertices, *V*. Each supplier *j* is modeled by an arc *j*, which is parameterized by an interval $\left[a_j, b_j\right]$.

Each processor *j* is characterized by a maximum processing rate μ_j, length L_j, and processing time T_j. The rate L_j / T_j describes the processing velocity. The dynamic of each processor on an arc *j* is governed by the following equation:

$$\partial_t \rho_j(x,t) + \partial_x \min\left\{\frac{L_j}{T_j}\rho_j(x,t), \mu_j\right\} = 0 \qquad (1)$$

$$\forall\, x \in \left[a_j, b_j\right],\; t \in \mathbb{R}^{+},$$

$$\rho_j(x,0) = \rho_{j,0}(x) \;\; \forall\, x \in \left[a_j, b_j\right], \qquad (2)$$

where $\rho_j(x,t)$ is the density of parts in the processor *j* at the point *x* and at time *t*.

The equation (1) has the following interpretation: parts are processed with a velocity L_j / T_j, but with a maximum allowed flux μ_j. If the incoming flux, for the arc *j*, becomes higher than the maximal allowed flux μ_j, a queue is formed.

At first, we analyze the case in which every vertex of the network, consisting of *N* suppliers, is connected exactly to one incoming arc and one outgoing arc, assuming that arcs are labeled in sequence, or the arc *j* is connected to the arc $j + 1$ and $b_j = a_{j+1}$. In this case, we are dealing with consecutive suppliers.

The queue associated to the supplier *j* is in front of each processor, or in $x = a_j$. Assume that the first supplier is characterized only by a processor with infinite length and that the last supplier has an infinite supplier, or $a_1 = -\infty$ and $b_N = +\infty$, respectively.

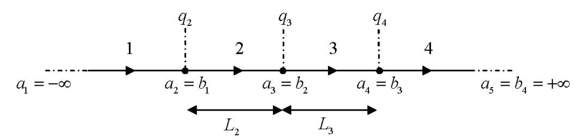

Figure 1: a scheme of a supply chain with $N = 4$.

The queue q_j for the processor *j* can be considered a function of time, or $q_j = q_j(t)$. Notice that there is no spatial dependence for the queue, as it is located at $x = a_j$. Variations of the length *j* are due to variations of fluxes for arcs $j - 1$ and *j*. Precisely, the queue grows if the flux for the arc $j - 1$ becomes higher than the flux for the arc *j*. Hence, the temporal variation of the queue, $\frac{d}{dt} q_j(t)$, satisfies the following differential equation:

$$\frac{d}{dt} q_j(t) = f_{j-1}\left(\rho_{j-1}\left(b_{j-1}, t\right)\right) - f_j\left(\rho_j\left(a_j, t\right)\right), \quad j = 2, \ldots, N. \qquad (3)$$

If the second term of (3) is positive, $q_j(t)$ starts to increase. Notice also that the fluxes for the arcs $j - 1$ and *j* are computed in the same spatial place that represents, as for the evolution of densities of arcs, a point where some discontinuities can occur.

The flux for the arc 1 of the supply chain is assigned and represents the incoming external flux. The remaining fluxes $f_j\left(\rho_j(x,t)\right)$, $j = 2, \ldots, N$, are given by:

$$f_j\left(\rho_j\left(a_j, t\right)\right) = \begin{cases} \min\left\{f_{j-1}\left(\rho_{j-1}\left(b_{j-1}, t\right)\right), \mu_j\right\}, & q_j(t) = 0, \\ \mu_j, & q_j(t) > 0. \end{cases} \qquad (4)$$

If the queue is empty, the flux of the arc $j-1$ can be processed on the arc j if such flux is lower than μ_j; otherwise, goods are processed at rate μ_j. If the queue is not empty, the flux $f_{j-1}\left(\rho_{j-1}\left(b_{j-1},t\right)\right)$ is higher than μ_j, hence goods are processed at rate μ_j.

Consider now the (DM) model. For exhaustive explanations and details, see D'Apice et al. 2006, D'Apice et al. 2008.

A supply chain consists of a sequence of $N+1$ sub-chains $I_1,\ldots, I_{N+1}$, and N suppliers or processors $P_1,...,P_N$ with certain throughput times and capacities. The supplier P_k connects the sub-chain I_k to the sub-chain I_{k+1}.

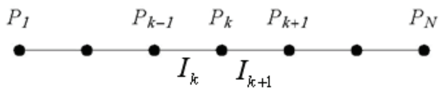

Figure 2: scheme of a supply chain.

Each supplier processes a certain good, measured in units of parts, and passes it in the next sub-chain. We assume that a node P_k consists of a processor, which decides how to manage the flow among sub-chains, with a maximal processing rate μ. Each sub-chain I_k is modelled by an interval $[a_k, b_k]$, with P_k corresponding to the coordinate b_k, on which we consider the system:

$$\begin{cases} \rho_t + f_\varepsilon\left(\rho,\mu\right)_x = 0, \\ \mu_t - \mu_x = 0. \end{cases} \tag{5}$$

For $\varepsilon > 0$, the flux function for each arc is defined in the following way:

$$f_\varepsilon^k\left(\rho,\mu\right) = \begin{cases} \rho, & 0 \le \rho \le \mu, \\ \mu + \varepsilon\left(\rho-\mu\right), & \mu \le \rho \le \mu_k^{\max}, \end{cases} \tag{6}$$

or alternatively

$$f_\varepsilon^k\left(\rho,\mu\right) = \begin{cases} \varepsilon\rho + \left(1-\varepsilon\right)\mu, & 0 \le \mu \le \rho, \\ \mu + \varepsilon\left(\rho-\mu\right), & \rho \le \mu \le \mu_k^{\max}, \end{cases} \tag{7}$$

with $\rho_k^{\max}$ and $\mu_k^{\max}$ the maximum density and processing rate on the arc I_k.

It is possible to generalize all following definitions and results to the case of different fluxes $f_{\varepsilon_k}^k$ for each line I_k (also choosing ε dependent on k). In fact, all statements are in terms of values of fluxes at endpoints of the sub-chains, thus it is sufficient that the ranges of fluxes intersect. Moreover, we can consider different slopes m_k for each line I_k, considering the following flux:

$$f_\varepsilon^k\left(\rho,\mu\right) = \begin{cases} m_k\rho, & 0 \le \rho \le \mu, \\ m_k\mu + \varepsilon\left(\rho-\mu\right), & \mu \le \rho \le \rho_k^{\max}, \end{cases} \tag{8}$$

where $m_k \ge 0$ represents the velocity of each processor and is given by $m_k = \frac{L_k}{T_k}$, with L_k and T_k, respectively, fixed length and processing time of processor k.

We interpret the evolution at nodes P_k thinking to it as a Riemann problem (a Cauchy Problem corresponding to an initial data which is constant on each supply line) for the density equation with processing rates as parameters. Riemann problems are solved fixing different rules, which conserve the flux at nodes. Here, we refer to the following one:

(SC) The objects are processed in order to maximize the flux. Then, if a solution with only waves in the density ρ exists, then such solution is taken; otherwise, the minimal μ wave is produced.

Rule (SC) corresponds to the case in which processing rate adjustments are done only if necessary, while the density can be regulated more freely. Thus it is justified in all situations in which processing rate adjustments require re-building of the supply chain, while density adjustments are operated easily (e.g. by stocking).

Notice that algorithm (SC) is appropriate to reproduce also the well known "bull-whip" effect, a well known oscillation phenomenon in supply chain theory, see Daganzo 2003.

3. NUMERICAL METHODS FOR SUPPLY CHAINS

In order to simulate the behaviour of the supply chains, numerical schemes ad hoc have been realized in order to discretize the equations describing the mathematical models.

Let us refer to the (GHK) model. In this case, two different numerical methods have been used, one for the conservation law describing the evolution of the density and the other for the linear ordinary differential equation describing the evolution of the queue. For the conservation law, an upwind scheme is the more suitable because of the flux function shape of each arc. We deal with a discrete grid in the plane $\left(x,t\right)$ based on the finite difference method. A temporal step Δt and a spatial step h are chosen and the grid points $\left(x_j,t^n\right)$ are defined as follows:

$$\begin{aligned} x_j &= jh,\ j \in \mathbb{Z}, \\ t^n &= n\Delta t,\ n \in \mathbb{N}. \end{aligned} \tag{9}$$

We set $\lambda = \Delta t / h$ (CFL condition, see Godunov 1959), and define $x_{j+1/2} = x_j + h/2$. The discrete solutions ρ_j^n, which approximate the density $\rho\left(x_j, t^n\right)$ for every j and n, for a generic arc of the supply chain, can be written as:

$$\rho_j^{n+1} = \rho_j^n - \lambda\left(H_{j+1/2}^n - H_{j-1/2}^n\right), \quad (10)$$

where H is the "numerical flux". For the upwind case, the numerical flux is equal to:

$$H_{j+1/2} = \frac{1}{2}\left[a\left(\rho_{j+1} + \rho_j\right) - a\left(\rho_{j+1} - \rho_j\right)\right], \quad (11)$$

where $a = \frac{L}{T}$, considering that L and T are referred to a generic arc, whose temporal evolution has to be computed.

In order to guarantee the convergence of the scheme, we choose $\left|\lambda \frac{L}{T}\right| \le 1$.

The Euler method can be used for the queues equation (3). For the arc j, the approximation of the queue $q_j\left(t\right)$ at time t^{n+1}, $q_j\left(t^{n+1}\right)$, is q_j^{n+1}, which is found recursively as:

$$q_j^{n+1} = q_j^n + \Delta t\left(f_{j-1}^n - f_j^n\right), \quad (12)$$

where f_{j-1}^n and f_j^n are, respectively, the approximation of fluxes $f_{j-1}\left(\rho_{j-1}\left(b_{j-1}, t^n\right)\right)$ and $f_j\left(\rho_j\left(a_j, t^n\right)\right)$.

Initial data are necessary for every queue, and every arc. Moreover, a boundary condition, given by the input profile $f_1\left(t\right)$ is needed for each arc (see Gottlich et al. 2005, Gottlich et al. 2006). Such profile can be translated into initial data $\rho_{1,0}\left(x\right) := \rho_{1,0}\left(b_1 - t\right) = f_1\left(t\right)$ on the first arc (assumed artificial), with the assumptions that $\mu_1 > \max f_1$, and $\frac{L_1}{T_1} = 1$.

Consider now the (DM) model. In this case, we refer to a Godunov method for a 2×2 system (details are in Bretti et al. 2007, Godunov 1959). Let Δt and Δx be, respectively, the temporal and the spatial step for the discrete grid in the plane $\left(x, t\right)$, whose points are $\left(x_j, t^n\right) = \left(j\Delta x, n\Delta t\right)$, $j \in \mathbb{Z}$, $n \in \mathbb{N}$; let ρ_j^n and μ_j^n be the approximations of $\rho\left(x_j, t^n\right)$ and $\mu\left(x_j, t^n\right)$, respectively. Then, the approximation scheme for (5) can be defined as follows:

$$\begin{cases} \rho_j^{n+1} = \rho_j^n - \frac{\Delta t}{\Delta x}\left(g\left(\rho_j^n, \rho_{j+1}^n\right) - g\left(\rho_{j-1}^n, \rho_j^n\right)\right), \\ \mu_j^{n+1} = \mu_j^n + \frac{\Delta t}{\Delta x}\left(\mu_{j+1}^n - \mu_j^n\right), \end{cases} \quad (13)$$

where the Godunov numerical flux g can be found solving Riemann problems among the states $\left(\rho_-, \mu_-\right)$ on the left and $\left(\rho_+, \mu_+\right)$ on the right:

$$g\left(\rho_-, \mu_-, \rho_+, \mu_+\right) = \\ = \begin{cases} \left(\rho_-, -\mu_+\right), & if\ \rho_- < \mu_- \ \vee\ \rho_- \le \mu_+, \\ \left(\frac{1-\varepsilon}{1+\varepsilon}\mu_+ + \frac{2\varepsilon}{1+\varepsilon}\rho_-, -\mu_+\right), & if\ \rho_- < \mu_- \ \vee\ \rho_- > \mu_+, \\ \left(\frac{1+\varepsilon}{2}\rho_- + \frac{1-\varepsilon}{2}\mu_-, -\mu_+\right), & if\ \rho_- \ge \mu_- \ \vee\ \mu_+ > \tilde{\mu}, \\ \left(\frac{1-\varepsilon}{1+\varepsilon}\left(\mu_+ + \varepsilon\mu_-\right) + \varepsilon\rho_-, -\mu_+\right), & if\ \rho_- \ge \mu_- \ \vee\ \mu_+ \le \tilde{\mu}, \end{cases} \quad (14)$$

with

$$\tilde{\mu} = \mu_- + \frac{1+\varepsilon}{2}\left(\rho_- - \mu_-\right). \quad (15)$$

For this numerical formulation of the model defined by (5), it is necessary to define the value of the boundary data, given by the term ρ_{j-1}^n. For the first arc of the supply chain, ρ_{j-1}^n is given by an input profile. If $m_1 = 1$, the boundary data is associated to the incoming external flux for the supply chain. Otherwise, ρ_{j-1}^n is determined by the solution to Riemann problems at nodes P_k.

Remark. The construction of the Godunov method is based on the exact solution to the Riemann problem (Bretti et al. 2007, Godunov 1959) in the cell $\left(x_{j-1}, x_j\right) \times (t^n, t^{n+1})$. To avoid the interaction of waves in two neighbouring cells before time Δt, we impose a CFL condition like:

$$\frac{\Delta t}{\Delta x}\max\left\{\left|\lambda_0, \lambda_1\right|\right\} \le \frac{1}{2}, \quad (16)$$

where λ_0 and λ_1 are the eigenvalues of the system (5). Since, in this case, the eigenvalues are such that $\left|\lambda_0\right| = 1$, $\left|\lambda_1\right| \le 1$, the CFL condition reads as: $\frac{\Delta t}{\Delta x} \le \frac{1}{2}$.

4. WORKING AREA OF THE TOOL FOR SUPPLY CHAINS

The tool for supply chains, that can be used either for the (GHK) model or the (DM) model, consists of three not coupled components:

- a calculus component in C++, for the implementation of the numerical methods associated to the equations of the supply chains models;
- a graphical component, realized in Java, that communicate with the calculus component, in order to reconstruct the profiles of densities and the behaviour of queues through animations;
- a web component, which allows the tool to work under some web applications.

Notice that, using the graphical tool, the user has not to worry about the insertion of parameters for the numerical approximation (namely spatial and temporal steps for the numerical approximation discrete grid), since such parameters are automatically chosen by the tool in order to establish the convergence of numerical schemes.

In what follows, we describe how it is possible to create the graph of a supply chain and then to simulate the goods dynamics, using the (GHK) model. First of all, it is necessary to introduce the initial node. Notice that such node is virtual as it does not belong to the topology of the real network, but indicates the incoming point of the external profile of goods. Then, the user can insert the other (real) nodes of the supply chain, that has to be simulated. All these operations can be done through a series of buttons, that are in the upper part of the working area.

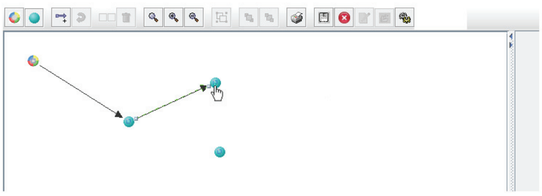

Figure 3: creation of an arc among two nodes.

For every arc and node, a menu is associated. The menu for nodes consists of only one feature: *remove*, useful to remove the selected node. The menu for the arcs is characterized by two features: *remove*, by which it is possible to clean the selected arc, and *properties*, by which it is possible to insert some characteristics for each arc (Figure 4):

- **Name:** name of the arc;
- **Incoming flux:** incoming flux (necessary as boundary data, although an external profile is always assigned);
- **Start Density:** initial density on the arc;
- **Max flux:** maximal flux for the arc;
- **Initial queue:** initial condition for the queue;
- **Length:** length of the arc;
- **Processing Time:** processing time;
- **% in the arc:** it indicates if the incoming flux can totally be directed to the outgoing arc; in supply chains, it is useful to choose it equal to one.

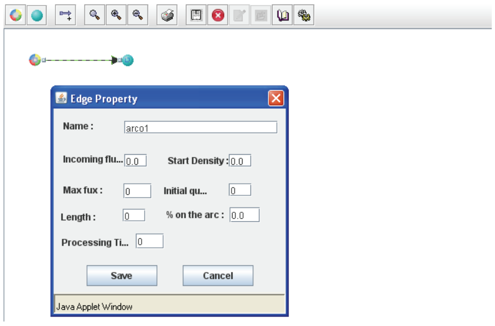

Figure 4: edge property window.

After the creation of the graph, in order to start the simulation we have to choose the simulation time. Then, the evolution of the density on the arcs and the evolution of the queues can be visualized. Different colours are associated to every interval for parts densities along the arcs of the supply chain, according to the table below.

Table 1: table of colours.

Interval of density	Colour
0 $\leq$ density < 20	
20 $\leq$ density < 40	
40 $\leq$ density < 60	
60 $\leq$ density < 80	
80 $\leq$ density < 100	
100 $\leq$ density < 120	
120 $\leq$ density < 140	
140 $\leq$ density < 160	
160 $\leq$ density < 180	
180 $\leq$ density $\leq$ 200	

5. SIMULATION RESULTS

We present some simulation results for the supply chain in the following figure:

Figure 5: Supply chain with 4 arcs and 3 (real) nodes.

Let us first refer to the (GHK) model. Notice that the first arc is not artificial, as it does not contain any queue, unlike the other arcs. The considered simulation starts from an initial condition of empty network, with empty queue. The total simulation time is $T = 190$.

Length, maximal flux for each arc and processing time, or L_j, μ_j, e T_j, respectively, for $j = 1,2,3,4$, are kept in the following table, where details for the considered supply chains are reported:

Table 2: parameters for the supply chain.

Arc j	μ_j	T_j	L_j
1	25	1	1
2	15	3	2
3	10	4	3
4	15	2	1

We assume an input profile of triangular shape, as in Figure 6:

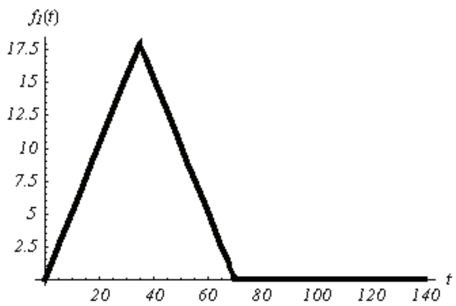

Figure 6: input profile for the firs arc of the supply chain.

In what follows, we report the state of goods traffic in some different instants of time for the analyzed supply chain, initially empty. Let us analyze Figure 7.

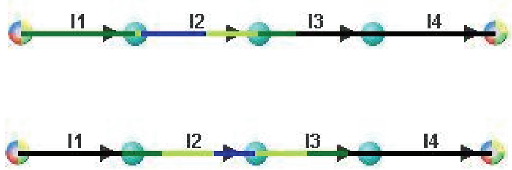

Figure 7: densities on arcs in time instants t = 70 (up) and t = 90 (bottom).

In Figure 7, at t = 70, the goods flux is quite low (it is evident from the colour of the first arc). A strong flux interests the second arc. The third arc is becoming to be full. At a first moment, the third arc is interested by a low goods flux. At t = 90, the readjustments of fluxes, because of queues at nodes, imposes a low flux on the first arc.

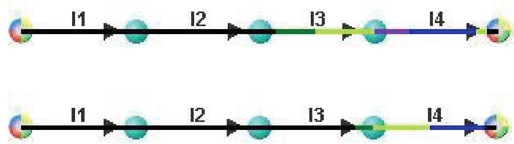

Figure 8: densities on arcs in time instants t = 130 (up) and t = 160 (bottom).

In Figure 8, at t = 130, the fourth arc is almost entirely full (violet and blue colours) and it is interested by a strong outgoing density. This is not surprising, since it means that there is a correct management of goods in the arcs 1, 2, and 3. Practically, the outgoing arc keeps all fluxes that have been re-elaborated by other arcs. At t = 160, the arc 4 starts to become empty.

In what follows, we consider also the temporal behaviour of queues, as you can see from Figure 9. In this case, the queue related to the arc four, $q_4(t)$ is zero, as $\mu_4 > \mu_3$. The different behaviours of queues $q_2(t)$ and $q_3(t)$ is essentially due to differences in processing velocity for arcs two and three. In particular, $v_2 = 0.66 < v_3 = 0.75$. We could expect that the arc three should process goods in a more faster way, but this does not occur since $L_3 > L_2$ and $T_3 > T_2$. This implies that, although $v_3 > v_2$, goods remain in the processor three for a greater time with respect to the processor two. Hence, $q_3(t)$ is higher than $q_2(t)$ and the needed time interval to let $q_3(t)$ become zero is higher than the necessary time interval for $q_2(t)$.

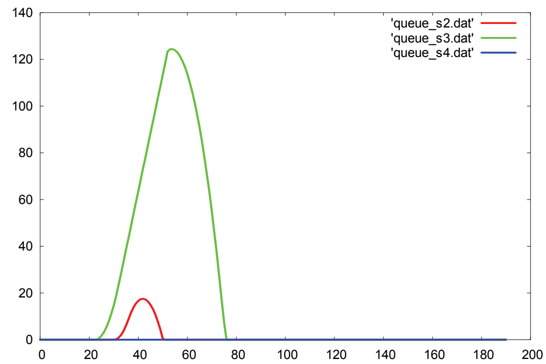

Figure 9: temporal behaviour for queues $q_2(t)$, $q_3(t)$ and $q_4(t)$, in red, green and blue, respectively.

Let us now to present some results obtained using the (DM) model. We consider a supply chain of $N = 4$ suppliers and impose the following initial and boundary data:

$$\rho_1(0,x) = \rho_2(0,x) = \rho_3(0,x) = \rho_4(0,x) = 0,$$
$$\rho_1(t,0) = \frac{\mu_2}{2}\left(1 + \sin\frac{3\pi t}{T_{\max}}\right), \quad (17)$$

where the space interval is [0,6] and the observation time is T_{max} = 20, with Δx=0.1 and Δt=0.05. On each processor, we assume that $\mu(0,x) = \mu_k$ and incoming and outgoing boundary data are given by μ_k. Notice that in this case the input profile $\rho_1(t,0)$ exceeds the maximum capacity of the processors.

We make simulations setting parameters as in the following table, assuming default processing velocities on each processor, namely $m_k = \frac{L_k}{T_k} = 1,\ k = 1,2,3,4$, where L_k is the length of the sub-chain and T_k the processing time:

Table 3: parameters for the supply chain.

Processor k	μ_k	L_k
1	99	1
2	15	1
3	10	3
4	8	1

In the next pictures, the evolution in time of flux, density and processing rate have been obtained using the rule (SC) for $\varepsilon = 0.1$.

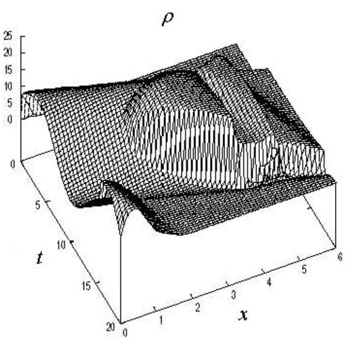

Figure 10: evolution of ρ.

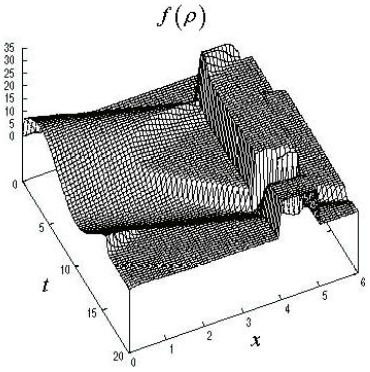

Figure 11: evolution of *f*.

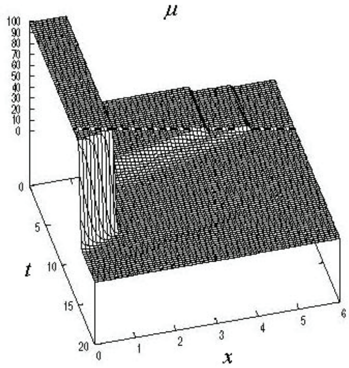

Figure 12: evolution of μ.

Since the algorithm (SC) maximizes the flux and adjusts the processing rate if necessary, minimizing its changes, we see that the processing rate of the first processor, initially equal to 99, in accordance to Table 3, is lowered to maximize the flux. The same happens with the other processors, even if the phenomena is less evident due to the initial values of the processing rate which are very low. Moreover we observer that the second processor is characterized by a great flow of goods.

From the analysis of the graphics we conclude that the algorithm (SC) gives rise to an interesting dynamics, which also fits with the (GHK) model. In particular, for ε tending to zero, the maximum values assumed by the flux and the density decrease.

CONCLUSIONS

In this paper, we described a tool able to simulate the behavior of the goods flow in a sequential supply chain using due different fluid dynamic models.

In future, we plan to integrate this tool with some optimization techniques in order to improve the performance of the supply in terms of a correct dimensioning of processing velocities and also of the processing rates for the (DM) model so as to reduce queues of production. Such study is highly not trivial, since one has to deal with the minimization of a numerical cost functional, which considers the lengths of queues in the case of the (GHK) model. This functional cannot be defined from an analytical point of view, as queues are evaluated numerically! Hence, the minimization problem requires the use of advanced numerical methods, which could be more expensive. The aim is elaborating fast algorithms, that could be able to achieve the required minimum with less computational times.

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